Comenius University, Bratislava

Faculty of Mathematics, Physics and Informatics

Extendability of matchings in graphs on surfaces

Bachelor Thesis

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Comenius University, Bratislava

Faculty of Mathematics, Physics and Informatics

Extendability of matchings in graphs on surfaces

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Study programme: Computer Science

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Supervisor: Mgr. Michal Kotrbčík

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Abstract

Graph G is said to be *equimatchable*, if every matching in G extends to (i.e., is a subset of) a maximum matching. In the paper [K. Kawarabayashi and M. D. Plummer. Bounding the size of equimatchable graphs of fixed genus. *Graphs and Combinatorics*, 25(1):91–99, 2009.] it is showed that for any fixed g, there are only finitely many 3-connected equimatchable graphs G embeddable in the surface of genus g with the property that either G is non-bipartite or the embedding has representativity at least three. The proof is based on a result that the maximum size of such a graph is at most $c \cdot g^{3/2}$, where c is a constant. In this thesis we show that the upper bound on the number of vertices of a 2-connected, non-bipartite, equimatchable graph embeddable in the surface of genus g is between $5\sqrt{g} + 6$ and $4\sqrt{g} + 17$ for any $g \leq 2$, between $5\sqrt{g} + 6$ and $12\sqrt{g} + 5$ for any $g \geq 3$, and between $5\sqrt{g} + 6$ and $8\sqrt{g} + 5$ for $g \geq 63$. Our methods are based on and refine the concept of isolating matchings used in the aforementioned paper. Moreover, we provide additional results concerning the structure of factor-critical equimatchable graphs and graphs embeddable in a fixed surface.

KEYWORDS: graph, graph embedding, genus, matching, equimatchable graph, surface.

Abstrakt

Graf G sa nazýva *equimatchable*, ak sa každé jeho párenie dá rozšíriť na najväčšie párenie v G; teda každé párenie je podmnožinou nejakého najväčšieho párenia. V článku [K. Kawarabayashi and M. D. Plummer. Bounding the size of equimatchable graphs of fixed genus. *Graphs and Combinatorics*, 25(1):91–99, 2009.] je dokázané, že pre ľubovoľné fixné g existuje iba konečne veľa trojsúvislých equimatchable grafov G vnoriteľných do plochy rodu g s vlastnosťou, že G je nebipartitný, alebo vnorenie má reprezentativitu aspoň tri. Dôkaz je založený na výsledku hovoriacom, že maximálny počet vrcholov takéhoto grafu je $c \cdot g^{3/2}$ pre nejakú konštantu c. Hlavným výsledkom tejto práce je tvrdenie, že maximálny počet vrcholov dvojsúvislého, nebipartitného, equimatchable grafu vnoriteľného do plochy rodu g je medzi $5\sqrt{g} + 6$ and $4\sqrt{g} + 17$ pre $g \leq 2$, medzi $5\sqrt{g}+6$ a $12\sqrt{g}+5$ pre ľubovoľné $g \geq 3$ a medzi $5\sqrt{g}+6$ a $8\sqrt{g}+5$ pre $g \geq 63$. Naše metódy sú založené na a ďalej spresňujú koncept izolujúcich párení využitých v uvedenej práci. Medzi ďalšie výsledky patrí štrukturálny popis faktorovo-kritických equimatchable grafov a grafov vnoriteľných do daných plôch.

KĽÚČOVÉ SLOVÁ: graf, párenie, vnorenie grafu, rod plochy, equimatchable graf.

Contents

Introduction

In this thesis we investigate equimatchable graph that can be embedded in a surface of a fixed genus. Equimatchable graphs are exactly the graphs in which one can always find maximum matching in linear time using greedy algorithm. Formally, a graph is called *equimatchable* if every its matching is a subset of a maximum matching. Equimatchable graphs with a perfect matching were characterized by Summer in [Sum79] and all such graphs are isomorphic to K_{2n} or $K_{n,n}$. A polynomial algorithm for verifying membership and non-membership in the class of equimatchable graphs can be found in [LPP84]. In the paper [KPS03] it is showed that there are precisely twenty-three 3-connected equimatchable planar graphs. Later, Kawarabayashi and Plummer in paper [KP09] showed that for any fixed g, there are only finitely many 3-connected equimatchable graphs G embeddable in the surface of genus g with the property that either G is non-bipartite or the embedding has representativity at least three. The proof is based on a result that the maximum size of such a graph is at most $c \cdot g^{3/2}$, where c is a constant.

In this thesis we focus mostly on equimatchable factor-critical graphs. We should note that every 2-connected equimatchable graph that is not bipartite is factor-critical. We provide several results characterizing the structure of 2-connected factor-critical equimatchable graphs that allow us to prove our main result:

Theorem. *Let $f(g)$ be function that gives the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the surface of orientable genus g. Then:*
i) If $g \leq 2$, then $5\sqrt{g} + 6 \leq f(g) \leq 4\sqrt{g} + 17$.
ii) If $g \geq 3$, then $5\sqrt{g} + 6 \leq f(g) \leq 12\sqrt{g} + 5$.
iii) If $g \geq 63$, then $5\sqrt{g} + 6 \leq f(g) \leq 8\sqrt{g} + 5$.

Additionally, we extend the results from [KPS03] by showing that when we allow graphs that are not 2-connected, there are infinitely many equimatchable planar connected graphs, both factor critical and bipartite.

1

Definitions and Preliminaries

This chapter is devoted to a presentation of the basic concepts used in this thesis. We start with a summary of used graph-theoretic notation. In the second part of this chapter we define matching, equimatchable graph, and related concepts, present Edmonds-Gallai decomposition theorem and a characterization of equimatchable graphs based on this decomposition. The rest of this chapter consists from a short foundation of topology and topological graph theory.

1.1 Graphs

First we present some basic notation and definitions used throughout the text, for the concepts not defined the reader is referred to [Die05].

A *graph* is a pair $G = (V, E)$ of sets such such that $E \subseteq [V]^2$; thus the elements of E are 2-elements subsets of V. The elements of V are the *vertices* (or *node*, or *points*) of the graph G, the elements of E are its *edges* (or *lines*). The usual way to picture graph is by drawing a dot for each vertex and joining two vertices by a line if the corresponding two vertices form an edge. The vertex and edge set of a graph G are also denoted by $V(G)$ and $E(G)$, respectively. Graphs in topological graph theory are usually with loops and multiple edges. Since, in our thesis we work exclusively with matchings, we could excludes loops and multiple edges in graphs (see Note 1.2 on page 4).

The number of vertices of a graph G is its *order*, written as $|G|$. The number of edges of graph G is denoted by $||G||$. A vertex v is *incident* with an edge e if $v \in e$; then e is an edge at v. The two vertices incident with an edge are its *endvertices* or *ends*. An edge $\{x, y\}$ is usually written as xy (or yx). Two vertices x, y are adjacent, or neighbours, if xy is an edge of G. Two edges $e \neq f$ are adjacent if they have a common vertex as their end. If all vertices of graph are pairwisely adjacent, then G is *complete*. A complete graph on n vertices is K_n. The *degree* $deg(v)$ of a vertex v is the number of edges incident with vertex v. By our definition the degree of a vertex v is equal to the number of neighbours of v. Set of neighbours of v is called *neighbourhood* (of v) and is denoted by $N(v)$. Let U be a set of vertices such that $U \subseteq V$. Then $N(U)$ denotes union of neighbourhoods of all vertices of U. If every vertex of the graph G has the same degree k, then G is said to be *k-regular*. A set of vertices or edges is said to be *independent* if no two of its elements are adjacent.

A graph is said to be *connected* if for any vertices a, b of G there is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices of graph such that $a = v_0$, $b = v_n$, and for each i the vertices v_i, v_{i+1} are adjacent. The maximal connected subgraphs of a graph G are called (connected) *components* of G. A graph is *k-edge-connected* for $k \geq 2$ if G is connected and for any set S of $k-1$ edges of G, the graph $G \setminus S$ is connected. Similarly, G is *k-vertex-connected*, or just *k-connected*, if it is connected and for every set S of $k-1$ vertices of G, the graph $G \setminus S$ is connected and it is not an isolated vertex. An edge e is a *bridge* if G is connected but $G \setminus e$ is not. Similarly, a vertex v is *cut-vertex* (or articulation) of G if G is connected but $G \setminus v$ is not.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is said to be a *subgraph* of graph G. If G' is a subgraph of a graph G such that $V' = V$ then G' is *spanning* subgraph of G. Subgraph $H = (V', E')$ of a graph $G = (V, E)$ is said to be *induced by* V' if for every edge $e \in E$ holds: if both ends of e are in V', then $e \in E'$. We denote the subgraph of graph G induced by vertex set U as $G[U]$, or just U, when it is clear that we mean a subgraph, not a vertex set.

Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is said to be *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must be independent. Instead of '2-partite' one usually says *bipartite*. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete* (multipartite). Complete r-partite graph with partitions of sizes $n_1, \ldots, n_r$ is denoted by $K_{n_1,\ldots,n_r}$. Bipartite graphs are characterized by the following well-known property:

Proposition 1.1.1. *A graph is bipartite if and only if it contains no odd cycle.*

1.2 Matchings

A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. Matching M is a matching of $U \subseteq V$ if every vertex of U is incident with an edge in M. The vertices in U are then called *matched* or *covered* (by M). Vertices not incident with an edge of M are *unmatched* or *uncovered*.

Note. In multigraphs, since a loop is considered to be adjacent to itself, they are banned to be in any matching. Only one edge between vertices u, v of graph G can be in matching. Therefore, for matchings it is important only if u and v are adjacent, and not how many edges are between them. Let a graph G be formed from a multigraph H by removing loops and replacing multi-edges by single edge. Then G has a matching M if and only if there exists a matching M' of H such that edge $xy \in M$ if and only if there is edge between vertices x and y in M'.

For a matching M, $|M|$ denotes the number of edges of M. A matching M in a graph $G = (V, E)$ is said to be *maximal* if any set $M' \subseteq E$, with $M' \supset M$ is not a matching in G. A matching M in G is *maximum* if, among all matchings in G, it is one with largest cardinality.

A k-regular spanning subgraph is called *k-factor*. Thus, a subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of $V(G)$. A non-empty graph $G = (V, E)$ is said to be *factor-critical* if G ha no 1-factor but for every vertex $v \in V$ the graph $G \setminus \{v\}$ has an 1-factor. A matching M that is an 1-factor is called *perfect* matching. If matching M leaves uncovered just one vertex, then M is said to be *near-perfect* matching.

The following theorem shows a necessary condition for bipartite graphs to have matching that saturates one partition.

Theorem 1.2.1 ([Hal35])**.** *Let G be bipartite graph with partitions A and B. Then G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

1.2.1 Equimatchable graphs

A graph in which every matching extends to (i.e., is a subset of) a perfect matching is said to be *randomly matchable*. More generally, a graph in which every matching extends to (i.e., is a subset of) a maximum matching is called *equimatchable*.

Randomly matchable graphs were already characterized by Summer in [Sum79].

Theorem 1.2.2 ([Sum79])**.** *A connected graph is randomly matchable if and only if* $G = K_{n,n}$ *or* $G = K_{2n}$.

Now we are ready to present Gallai-Edmonds (D, A, C) decomposition, which is very useful in the study of matchings in graphs, in particular in the study of equimatchable graphs.

For a graph $G = (V, E)$ denote by D the set of all vertices of G which are not saturated by at least one matching of G. Let A be the neighbor set of D, i.e., the set of vertices in $V - D$ adjacent to at least one vertex in D. Finally $C = (V - D) - A$. Then (D, A, C) is called *Gallai-Edmonds decomposition* of the graph G. Using Gallai-Edmonds decomposition the following theorem describes the structure of all maximum matchings in graph G. The theorem was proved independently by Gallai ([Gal63], [Gal64]) and Edmonds ([Edm65]).

Theorem 1.2.3 (Gallai-Edmonds Structure Theorem [Gal63, Gal64, Edm65])**.** *Let G be a graph and (D, A, C) its Gallai-Edmonds decomposition. Then all the following conditions hold:*

(i) the components of the subgraph induced by D are factor-critical;

(ii) the subgraph induced by C has an 1-factor;

(iii) if M is a maximum matching of G, it contains a near-perfect matching of each component of D, a 1-factor of each component of C, and matches all vertices of A with vertices in distinct components of D;

(iv) the bipartite graph obtained from G by deleting the vertices of C and edges spanned by A and by contracting each component of D to a single vertex has a matching that saturates A.

(v) The size of any maximum matching is $\frac{1}{2}(|V| - \omega(D) + |A|)$*, where* $\omega(D)$ *is the number of componenents of* $G[D]$*.*

Using the previous theorem it is easy to prove the next lemma stated as Lemma 1 in [LPP84].

Lemma 1.2.4. *Let* G *be a connected equimatchable graph with no perfect matching, having Gallai-Edmonds decomposition* (D, A, C)*. Then* $C = \emptyset$ *and* A *is an independent set in* G*.*

The following characterization of equimatchable graphs was proved in [LPP84].

Theorem 1.2.5 ([LPP84])**.** *Let* G *be a connected equimatchable graph without a perfect matching. Let* (D, A, C) *be its Gallai-Edmonds decomposition and suppose* $A \neq \emptyset$*. Let* D_i *denote any component of* D *with* $|D_i| \geq 3$*. Then all of the following conditions hold:*

(1) Component D_i *must be one of following types of graphs:*

I. *$D_i \cong K_{2m+1}$ for some $m \geq 2$ and every point of D_i is joined to exactly one common point $a \in A$.*

II. *D_i contains a cut-vertex d_i of G (called* hook *of D_i) which is the only vertex of D_i adjacent to a point of A. Let $H_i^1, \ldots, H_i^r$ be the components of $D_i - d_i$. Consider any one of these, say H_i^j. There are two possibilities:* (a) *$H_i^j \cong K_{2m}$ for some $m \geq 1$ and at least two edges join d_i to H_i^j, or* (b) *$H_i^j \cong K_{m,m}$ for some $m \geq 1$ and if (U, W) is the bipartition of H_i^j, at least one edge joins d_i to a vertex u of U and at least one edge joins d_i to a vertex w of W.*

III. *At least two vertices of D_i are adjacent to points of A and at least one vertex of D_i is adjacent to no point of A. In this case there is a vertex $a \in A$ such that a separates D_i from rest of graph. Here we have four subcases. If D_i contains exactly two vertices y_1 and y_2 of attachment to a, then D_i must be one of following three types:* (a) *D_i is K_3;* (b) *$(D_i - y_1 - y_2)$ is a complete bipartite graph $Kr, r-1$ where $r \geq 2$, and if (U, W) is the bipartition of $D_i - y_1 - y_2$ where $|U| = r$, then y_1 and y_2 are both adjacent to all points of U and to each other;* (c) *$(D_i - y_1 - y_2)$ is K_{2r-1}, $r \geq 2$, y_1 and y_2 are both adjacent to all vertices of $D_i - y_1 - y_2$ and*

y_1 and y_2 may or may not be adjacent to each other. The fourth subcase may be stated as follow: (d) *if D_i has between* 3 *and $|D_i| - 1$ points of attachment to a, then D_i is K_{2r-1} for some $r \geq 3$.*

(2) Suppose we delete all type II and type III components of D from G and contract all type I components to single points. Then there is a matching of resulting (bipartite) graph G' which covers all vertices of A and G' is equimatchable.

The next theorem, converse of Theorem 1.2.5 was proved in [LPP84].

Theorem 1.2.6. *Let G be connected graph without a perfect matching, which is not factor-critical and which has Gallai-Edmonds decomposition (D, A, C). Suppose*
(1) $C = \emptyset$; and
(2) A is independent set.; and
(3) All components of D are singletons or of types I, II, or III as described in Theorem 1.2.5.; and
Let G_1 be the bipartite graph obtained from G by shrinking (contracting) all components of D to singletons and let G_1' be the graph obtained from G_1 by deleting all points corresponding to type II and III components of D. Suppose:
(4) G_1' is equimatchable graph and $|A| \leq \frac{1}{2}|V(G_1')|$.

Then G is equimatchable.

1.3 Surfaces and Embeddings

In this part we briefly introduce basic concepts of topological graph theory - topological surfaces and embeddings of graphs in surfaces. Most of definitions and theorems in this section is from [GT87] and from [Whi01]. For a deeper account of topology, the reader is referred to [Cro05].

An embedding of a graph in a surface generalizes the concept of an embedding of a graph in the plane. From a visual point of view, we can imagine embedding as a drawing of the graph on a sphere, torus, double-torus or a similar surface.

Formally, any graph can be presented by a topological space in following sense. Each vertex is represented by a distinct point and each edge by a distinct arc, homeomorphic to a closed interval $[0, 1]$. Naturally, the boundary points of an arc represent the ends of the corresponding edge. (Of course, interiors of arcs are mutually disjoint and do not meet the points representing vertices.) Such a space is called *topological representation* of the graph G.

Graphs G and H are said to be *homeomorphic* if they have respective subdivisions G' and H' such that G' and H' are isomorphic.

The central concern of topological graph theory is the placement of graphs on surfaces. A topological space M is called *n-manifold* if M is Hausdorff(see [Cro05]) and can be covered by countably many open sets, each of which is homeomorphic either to the n-dimensional open ball

$$\{(x_1, \ldots, x_n)|x_1^2 + \cdots + x_n^2 < 1\}$$

or the n-dimensional half-ball

$$\{(x_1, \ldots, x_n)|x_1^2 + \cdots + x_n^2 < 1, x_n \geq 0\}.$$

A manifold is *closed* if it is compact and its boundary is empty. By surface we usually mean closed, connected 2-manifold, such as the sphere, the torus, or the Klein bottle.

We define an embedding of a graph in a surface. Let G be a graph and S a surface. An *embedding* is a continous one-to-one function $\Pi : G \to S$. Usually, we consider our graphs to be subsets of the surface S, and the function $\Pi : G \to S$ is inclusion map. The embedding is then denoted simply $G \to S$.

Given an embedding $G \to S$, the components of $S - G$ are called *regions*. Regions are also called *faces* of embedding. If each region is homeomorphic to an open disc, the embedding is said to be 2*-cell (or cellular) embedding*. The closure in surface S of a region in the 2-cell embedding $G \to S$ need *not* be homeomorphic to closed disc. If there exists a boundary walk containing vertices x and y, then we say that vertices x and y are on same face of embedding $G \to S$.

Each face of an embedding $G \to S$ has two possible directions for its boundary walk. A face is assigned an *orientation* by choosing one of these two directions. An *orientation of embedding* $G \to S$ is an assigment of orientations to all faces so that adjacent regions induce opposite direction on every common edge. If a graph G is 1-skeleton of a triangulation of surface S, then orientation of embedding $G \to S$ is called *orientation of triangulation*. A surface is *orientable* if for every graph G there exists an embedding $G \to S$ with an orientation. If every embedding of a graph to a surface does not have an orientation, then the surface is said to be *non-orientable*. In this work, we will deal exclusively with orientable surfaces.

Given an orientable surface, we can add *handle* to it in such a way that the resulting object is an orientable surface. For example, we can obtain the torus by adding a handle to the sphere. In general, starting with the sphere S_0 we can add g handles to it. The resulting surface is called a sphere with g handles and it is denoted S_g . The number g is then called the *orientable genus* of the surface. The following crucial theorem asserts that these are essentially the only orientable surfaces.

Theorem 1.3.1. *The surfaces S_g, $g = 0, 1, 2, \dots$ are pairwise non-homeorphic and every closed orientable surface is homeomorphic to one of them.*

The minimum g such that there exists embedding $G \to S_g$ is called *genus of graph* and is denoted $\gamma(G)$. The maximum such g that there exists cellular embedding $G \to S_g$ is denoted $\gamma_M(G)$.

Theorem 1.3.2 ([Duk66])**.** *A connected graph G has a 2-cell embedding in S_g if and only if $\gamma(G) \le g \le \gamma_M(G)$.*

In our thesis we will use the following theorems about genus of complete and complete bipartite graphs frequently. All theorems can be found in chapter 6 of [Whi01].

Theorem 1.3.3 ([Rin65])**.** *Let $G = K_{m,n}$, with $m, n \ge 2$. Then*

$$\gamma(G) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Theorem 1.3.4 ([RY68])**.** *Let $G = K_n$, with $n \ge 3$. Then*

$$\gamma(G) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil .$$

In addition, we mention the next theorems about the maximum genus of complete and complete bipartite graphs.

Theorem 1.3.5 ([NSW71]). *Let $G = K_{m,n}$. Then*

$$\gamma_M(G) = \left\lfloor \frac{(m-1)(n-1)}{2} \right\rfloor.$$

Theorem 1.3.6 ([Rin72]). *Let $G = K_n$. Then*

$$\gamma_M(G) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor.$$

Let $\Pi : G \to S$ be an embedding. Denote number of vertices of G p, nubmer of edges q and number of faces in embeding Π as r. From this time forth in this section we will be using former denotation for number of vertices, edges and faces in Π.

Let Π be an embedding of a connected graph into a closed, connected surface. The *Euler characteristic of* Π is the value $p - q + r$, and it is denoted $\chi(\Pi)$. The following famous formula shows that for every standard surface the value of Euler characteristic is independent from the choice of graph and of a cellular embedding.

Theorem 1.3.7 (The Euler-Pointcaré formula). *Let $G \to S$ be a 2-cell embedding, for any $g = 0, 1, 2, \ldots$. Then $\chi(G \to S) = 2 - 2g$.*

The Euler-Pointcaré formula is often used in conjuction with relationship between the numbers of edges and faces to prove that certain graphs cannot be embedded into the surface S_g.

Theorem 1.3.8. *Let $\Pi : G \to S$ be an embedding of connected simple graph with at least three vertices into any surface. Then $2q \geq 3r$.*

Proof. The sum $\sum_{f\in F} s_f$, where s_f is number of sides of region f, counts every edge exactly twice. Thus, $2q = \sum_{f\in F} s_f$. Since there are no loops or multiple edges in simple graph G, there are no monogons or digons in the embedding. Therefore, for every region f, $s_f \geq 3$. It follows that $2p \geq 3r$. □

Actualy, if we have a graph with given girth then following theorem holds:

Theorem 1.3.9. *Let G be connected graph that is not a tree and let $\Pi : G \to S$ be an embedding. Then $2q \geq girth(G) \cdot r$.*

1.3.1 Rotantion systems

Define a 1-*band* to be a topological space b together with homeomorphism $h : I \times I \to b$, where I denotes the unit interval $[0, 1]$. The arcs $h(I \times \{j\})$ for $j = 0, 1$ are called *ends* of band b and the arcs $h(\{j\} \times I)$ for $j = 0, 1$ are called *sides* of band b. A 0-*band* and a 2-*band* are simply homeomorphs of the unit disc. A *band decomposition* of the surface S is collection B of 0-bands, 1-bands, 2-bands satysfying these conditions:

(1) Different bands intersect only along arcs in their boudaries.
(2) The union of all the bands is S
(3) Each end of each 1-band is contained in a 0-band.
(4) Each side of each 1-band is contained in a 2-band.
(5) The 0-bands are pairwise disjoint and the 2-bands are pairwise disjoint.

Corresponding *reduced band decomposition* B omits the 2-bands. Note that, in embedding $G \to S$ 0-bands represents vertices of G, 1-bands represents its edges and 2-bands represents regions of embedding. To describe a embedding $G \to S$ or equivalently its band decomposition are 2-bands not really needed to define, since the union of 1.bands and 0-bands is surface with boundary, and since is essenitally only one way how to fill in the faces to complete to closed surface.

A band decomposition is called *locally oriented* if each 0-band is assigned an orientantion. Then 1-band is called *orientation-preserving* if direction induced on its ends by adjoining 0-bands are the same as those induced by one of two possible orientation of 1-band; otherwise 1-band is called *orientation-reversing*. An edge e in graph embedding associated with localy oriented band decomposition is said to have *(orientation) type 0* if its corresponding 1-band is orientation-preserving and *(orientation) type 1* otherwise.

To describe a graph embedding $G \to S$ or equivalently its band decomposition, we need to specify only how the ends of 1-bands are attached to the 0-bands. We define *rotation* at a vertex v of graph to be ordered list, unique up to cyclic permutation, of the edges incident on that vertex. Let a *rotation system* on a graph be an assigment of a rotation to each vertex and a designation of orientation type for each edge. Then the preceding discusion can be summarized by following theorem.

Theorem 1.3.10. *Every rotation system on a graph G defines (up to equivalence of embeddings) a unique localy oriented graph embedding $G \to S$. Conversely, every locally* *oriented graph embedding $G \to S$ defines a rotation system for G.*

From now on we will use the terms embedding and rotation system interchangeably.

Given a rotation system for a graph, one frequently needs to obtain a listing or enumeration of boundary walks of the reduced faces. We first introduce some helpfull terminology. If rotation at vertex v is $\ldots de \ldots$, then we say d is *the edge before e at v*, that e is *the edge after d at v*, and that edge pair (d, e) is *corner at v with second edge e*.

To enumerate boudary walks of reduced faces we use following algorithn *Face Tracing Algorithm.*

Face Tracing Algorithm
Assume that given graph G has not any vertex of degree two.
(1) Choose an inicial vertex v_0 of G and a first edge e_1 incident on v_0. Let v_1 be the other endpoint of e_1.
(2) If the walk traced so far ends with edge e_i at vertex v_i then the next edge e_{i+1} in the boundary walk is the edge after (resp., before) e_i at v_i if e_i is type 0 (resp., type 1).
If the next two edges in the walk would not be e_1 and e_2 then
(3) Go to step (2).
Else
(4) The boundry walk is finished at edge e_n.
(5) If there is a corner at any vertex v that does not appear in any previously traced faces, then choose as initial vertex v and as the first edge second edge of this corner at v, and go to step (2)
(6) If there are not unused corners, then all faces have been traced.

Suppose graph G has some vertices with degree 2. Then we just find the graph H, without valent 2 vertices, such that G is subdivision of H. Then we use face-tracing algorithm on H and subdivide edges to correspond with graph G.

2

Previous work

In the paper [KP09] showed that for surface Σ of any fixed genus (orientable or non-orientable), there are only finitely many 3-connected equimatchable graphs embeddable in Σ that are non-bipartite or have a minimum-genus embedding of representativity at least three. In this chapter we provide a brief summarization of their proof.

First, we define the *Euler contribution* $\phi(v)$ of vertex v to be

$$\phi(v) = 1 - \frac{deg(v)}{2} + \sum_{i=1}^{deg(v)} \frac{1}{f_i},$$

where the sum is taken over all the face angles (corners, as defined on page 12) at vertex v and f_i denotes the size of the i-th face at v. Note that a face may contribute more than one face angle at a vertex v.

Lemma 2.1. *If a connected graph G is cellularly embedded in the surface of orientable genus g (resp. non-orientable genus $\bar{g}$), then $\sum_v \phi(v) = 2 - 2g$ (resp. $2 - \bar{g}$).*

Proof. Denote number of vertices of G by p, number of edges by q and number of faces by r. Clearly,

$$\sum_v \phi(v) = \sum_v \left(1 - \frac{deg(v)}{2} + \sum_{i=1}^{deg(v)} \frac{1}{f_i}\right) = p - q + \sum_v \sum_{i=1}^{deg(v)} \frac{1}{f_i}.$$

Since, every face has f_i face angles and every face angle is at exactly one vertex. Therefore, in sum $\sum_v \sum_{i=1}^{deg(v)} \frac{1}{f_i}$ is ith face counted exactly f_i times. Hence, $\sum_v \sum_{i=1}^{deg(v)} \frac{1}{f_i} = r$

and

$$\sum_v \phi(v) = p - q + r.$$

The lemma follows from Euler-Pointcaré formula (see Theorem 1.3.7 on page 10). □

2 Vertex-isolating Matchings in Embedded Graphs

Given a vertex $v \in V(G)$, a matching $M \subseteq E(G)$ is said to isolate v if M covers $N(v)$, but not v. In particular, we also say that M isolates v if $G \setminus (\{v\} \cup V(M))$ is empty.

The following was proved as **Theorem 2.1** in [KP09].

Theorem 2.1.1 ([KP09]). *Suppose G is a 3-connected graph of orientable genus g with $|V(G)| > max\{8, 24g - 24\}$ or of non-orientable genus $\bar{g}$ with $|V(G)| > max\{8, 12\bar{g} - 24\}$. Then all the following conditions holds:*

(i) $3 \leq \delta(G) \leq 6$*; and*

(ii) if $\delta(G) = 3$, for every vertex $v \in V(G)$ with $deg(v) = 3$ there is a matching $M_v \subset E(G)$ with $|M_v| \leq 3$ which isolates v; and

(iii) if $4 \leq \delta(G) \leq 6$, then for every vertex $v \in V(G)$ such that $deg(v) = \delta(G)$, there is either a matching $M_v \subseteq E(G)$ with $|M_v| \leq 4$ which isolates v or a neighbour of v.

Sketch of a proof. We follow the proof in [KP09]. Since orientable and nonorientable cases are analogous we present a proof of only the orientable case.

(i) Since G is 3-connected, we have $\delta(G) \geq 3$. The fact that $\delta(G) \leq 6$ follows from Euler's theorem (see Theorem 1.3.7 on page 10) and fact that $|V(G)| > max\{8, 24g - 24\}$.

To show (ii) and (iii) we will show that either there is vertex v in G and matching M_v with $|M_v| \leq 4$ isolating v or a neighbour of v or else we get contradiction with Lemma 2.1. The proof in [KP09] is by case-analysis, we list all cases and provide the proof of the first case.

Case 1: $\delta(G) = 3$ and v is a vertex of degree 3.

Case 2: $\delta(G) = 4$ and v is a vertex of degree 4. *Case 2* has following subcases. *Case 2.1:* There are at least three triangles at v (i.e., there are at least three edges between neighbours of v). *Case 2.2:* There are exactly two triangles at v. *Case 2.3:* There are exactly one triangle face at v. *Case 2.4:* There are no triangles at v.

Case 3: $\delta(G) = 5$ and v is a vertex of degree 5. Let the neighbours of v be $x_1, \ldots, x_5$. Suppose there is a triangle at v, say without loss of generality, vx_1x_2v. *Case 3.1:* There is edge between vertices x_3, x_4, and x_5; or *Case 3.1:* There is not edge between vertices x_3, x_4, and x_5.

Case 4: $\delta(G) = 6$ and v is a vertex of degree six and every face incident with vertex v is triangular.

Case 5: $\delta(G) = 5$ and no vertex of degree 5 there is incident with a triangle; or $\delta(G) = 6$ and every vertex of degree 6 is incident with non-triangular face.

Proof of the Case 1: Let the three neighbours of v be x_1 , x_2 and x_3.

Case 1.1: First, assume that there is at least one triangle at v. WLOG, denote this triangle by vx_1x_2v. If x_3 has a neighbor $y \notin \{v, x_1, x_2\}$, then $M_v = \{x_1x_2, x_3y\}$ is a matching of size 2 which isolates vertex v. So suppose there is no such y. Then either $G \cong K_4$ or x_1, x_2 is 2-cut. Since G is 3-connected and has at least $max\{8, 24g - 24\}$ vertices, both situations lead into a contradiction.

Case 1.2: Suppose that there is no triangle at v. Since $\delta(G) \geq 3$, there is a vertex $x_4 \notin \{v, x_1, x_2, x_3\}$ which is adjacent to x_1. Since x_2 is adjacent to neither x_1 nor x_3, choose a vertex $x_5 \in N(x_2) \setminus \{v, x_1, x_3, x_4\}$. If x_3 has a neighbor $y \notin \{v, x_1, x_2, x_4, x_5\}$, then $M_v = \{x_1x_4, x_2x_5, x_3y\}$ is a 3-matching (i.e., $|M_v| = 3$) isolating vertex v. So suppose that $N(x_3) = \{v, x_4, x_5\}$. If $N(x_1) = N(x_2) = \{v, x_4, x_5\}$, then either $\{x_4, x_5\}$ is 2-cut or $|V(G)| = 6$. Both of these situation result in a contradiction. Without loss of generality, let x_2 has a neighbor $y \notin \{v, x_1, x_3, x_4, x_5\}$. But then $M_v = \{x_1x_4, x_2y, x_3x_5\}$ is a 3-matching isolating v. $\square$

2.2 Equimatchable graphs of fixed genus

In this section we present a brief summary of the the main result from [KP09]. We start with a lemma needed in the proof.

Lemma 2.2.1 (Property 2. of [Fav86]). *Let G be a connected, equimatchable, non-bipartite graph that is neither factor-critical, nor randomly matchable. Then G has at least one cut-vertex.*

Proof. Let G be not factor-critical, non-bipartite graph with Gallai-Edmonds decomposition (D, A, C). The fact that G is not factor critical and does not have a perfect matching imply that $A \neq \emptyset$. From Theorem 1.2.6 on page 7 follows that D has at least one component of type I, II, or III, as described in Theorem 1.2.5 on page 6. Clearly, any component of type II has a cut-vertex. There is at least one singleton or type I component adjacent to vertex $a \in A$ separating component of type III from the rest of graph. Therefore, if there is a component of type III, then G has a cut-vertex. Any component of Type I is K_{2n-1} and, by Theorem 1.2.5, has all vertices connected to exactly one vertex a of A. Since G is not randomly matchable, a is cut-vertex. □

Using Theorem 2.1.1 from the previous section we present a proof of main result in [KP09].

Theorem 2.2.2 ([KP09]). *Let G be a 3-connected equimatchable graph of genus g (respectively, non-orientable genus $\bar{g}$). Then if G is non-bipartite or if G is bipartite and the representativity of the embedding is at least three, then*
$|V(G)| \leq max\{f_1(g), f_2(g), f_3(\bar{g}), f_4(\bar{g})\}$, *where*

$$f_1(g) = \left(\frac{7+\sqrt{1+48g}}{2}\right)\binom{8}{3}(4g+3)+9, \tag{2.1}$$

$$f_2(g) = 4\left(1+\sqrt{g}\right)\binom{8}{3}(4g+3)+9, \tag{2.2}$$

$$f_3(g) = \left(\frac{7+\sqrt{1+24\bar{g}}}{2}\right)\binom{8}{3}(2\bar{g}+3)+9 \tag{2.3}$$

and

$$f_4(g) = \left(4+2\sqrt{2\bar{g}}\right)\binom{8}{3}(2\bar{g}+3)+9. \tag{2.4}$$

Sketch of a proof. Similarly as for Theorem 2.1.1, orientable and nonorientable cases are analogous and we present only the orientable case.

Randomly matchable graphs are $K_{n,n}$ and K_{2n}. For any graph G from these classes, if G is embeddable in the surface of genus g then the number of vertices of G is lower than the maximum of functions $f_1(g)$ and $f_2(g)$ (see Theorem 1.3.3 and Theorem 1.3.4 on page 9).

From Lemma 2.2.1 follows that any 3-connected equimatchable graph without a perfect matching is either factor-critical or bipartite. Since a factor-critical graph cannot be bipartite, these cases are disjoint.

First, suppose G *is bipartite.* Since G is connected, from Edmonds-Gallai decomposition of G follows that the bipartition of G is (A, D). Clearly, every component of D is singleton. Since G has representativity at least three and G is 3-connected by Proposition 5.5.12 of [MT01] there is a cycle C_a in G covering $N(a)$, but not a. Moreover, cycle C_a has even length since G is bipartite. Choose every second edge of the cycle C_a to form a matching M_a isolating a. Extend M_a to maximal matching M. We have a maximal matching that that leaves a uncovered. This contradicts the fact that a is in A. Therefore, A is empty and G cannot be bipartite, implying there is no 3-connected bipartite equimatchable graph embedded with representativity at least three.

Second, suppose G *is factor-critical.* If $|V(G)| \leq max\{8, 24g - 24\}$, then the theorem clearly holds. Therefore, suppose $|V(G)| > max\{8, 24g - 24\}$. From Theorem 2.1.1 there exists a vertex v such that $deg(v) \leq 6$ and there exists a matching M_v with at most 4 edges that isolates vertex v from the rest of graph. Let G' be defined by $G' = G \setminus (V(M_v) \cup \{v\})$. Clearly, M_v has at most 8 vertices. Suppose G' has more than $\binom{8}{3}(4g + 3)$ components. Since G is 3-connected, using Pidgeon-hole principle it is easy to show that G contains a minor isomorphic to $K_{3,4g+3}$ and thus G is not embeddable in the surface of genus g. Therefore, G' has at most $\binom{8}{3}(4g+3)$ components. Since G is equimatchable and factor-critical, G' is randomly matchable. Clearly, every component of G' is randomly matchable, hence every component of G' is K_{2n} or $K_{n,n}$ and the theorem holds. $\square$

3

Equimatchable factor-critical graphs of fixed genus

This chapter is devoted to the study of equimatchable graphs embeddable to the surface with focus on 2-connected factor-critical equimatchable graphs. We provide several results characterizing the structure of factor-critical equimatchable graphs and graphs embeddable to the surface of fixed genus that allow us bound the number of vertices of 2-connected factor-critical equimatchable graphs embeddable to the surface of fixed genus.

Lemma 3.1. *Let N be a nonnegative integer.*

i) There is a connected planar factor-critical equimatchable graph with at least N vertices.

ii) There is a connected planar equimatchable graph with at least N vertices that is not factor-critical.

Proof. i) Let G be a graph formed from N triangles, choosing one vertex from every triangle and identifying the chosen vertices into one vertex. It is easy to see that graph G has exactly $2N+1$ vertices, is equimatchable and factor-critical planar graph.

ii) We construct the desired graph G as follows. The set of vertices $V(G)$ is $\{x_1, \ldots, x_N\} \cup \{y_1, \ldots, y_{2N+1}\}$ and only edges in $E(G)$ are the edges $x_i y_{2i-1}$, $x_i y_{2i}$, and $x_i y_{2i+1}$ for $i = 1, \ldots, N$. It is easy to see that graph G has exactly $3N+1$ vertices, is equimatchable and bipartite, hence not factor-critical, planar graph. □

Lemma 3.2. *Let G be a factor-critical graph. For every vertex $v \in V(G)$ there is a matching $M_v \subset E(G)$ with $|M_v| \leq deg(v)$ which isolates v.*

Proof. Since G is factor critical, the graph $G' = G \setminus \{v\}$ has a perfect matching M'. As long as G is simple, vertex v has $deg(v)$ neighbours. Clearly, every neighbour of v is incident to exactly one edge of matching M'. Consider a set $N \subseteq M'$ such that N contains precisely those edges from M' that are incident with at least one neighbor of v. Then N is desired matching M_v with at most $deg(v)$ edges that isolates v. □

When we say that a subgraph H_1 (such as a vertex, edge, or component) of a graph G is *linked* with other subgraph H_2 of same graph G we mean that there are vertices $k_1 \in H_1$ and $k_2 \in H_2$ such that $k_1 k_1 \in E(G)$.

Theorem 3.3. *Let G be a 2-connected, factor-critical equimatchable graph. Let $v \in V(G)$ be a vertex of G and M_v minimal matching that isolates v. Let $G' = G \setminus (V(M_v) \cup \{v\})$. Then G' is isomorphic with K_{2n} or $K_{n,n}$ for some nonnegative integer n.*

Proof. We prove the theorem by a series of claims.

Claim 1. Every component of G' is either K_{2n} or $K_{n,n}$.

Proof of Claim 1. Let M' be a maximal matching of G'. Clearly, $M = M' \cup M_v$ is a maximal matching of G. But G is factor-critical and equimatchable, therefore M leaves only vertex v uncovered and M' must be a perfect matching of G'. Since arbitrary maximal matching M' of G' is perfect matching of G', G' is randomly matchable and all of its components are either K_{2n} or $K_{n,n}$.

Claim 2. If xy is an arbitrary edge of matching M_v, then x and y cannot be linked to different components of G'.

Proof of Claim 2. We prove the claim by contradiction. Let C_1 and C_2 be different components of G' and suppose that x is adjacent to $x' \in C_1$ and y is adjacent to $y' \in C_2$. Let M be matching defined by $M = (M_v \setminus \{xy\}) (\cup \{xx'\} \cup \{yy'\})$. Clearly, every maximal matching $M' \supseteq M$ leaves uncovered vertex v (because every vertex linked to v is already in matching). From Claim 1 follows that C_1 and C_2 have even number of vertices and therefore $C_1 \setminus \{x'\}$ and $C_2 \setminus \{y'\}$ have odd number of vertices. Therefore, M' leaves also one vertex of C_1 and C_2 uncovered. This is contradiction to the fact that G is equimatchable and factor-critical.

Claim 3. Let C be a component of G' and xy be an edge of matching M_v, such that x is linked to some vertex $x' \in C$ then y is linked either to v or to some vertex $y' \in C$, with $y' \neq x'$.

Proof of Claim 3. We prove the claim by contradiction. Let y be not linked to either C or v. Let M be a matching defined by $M = (M_v \setminus \{xy\}) \cup \{xx'\}$. Clearly, every maximal matching $M' \supseteq M$ leaves uncovered vertex v. Since by Claim 2 y cannot be linked to any other component of G', y is not linked to v or C and matching M has yet covered every other vertex of M_v then M' leaves also y uncovered. Therefore, M' leaves at least two vertices uncovered and it is contradiction to the fact that G is equimatchable and factor-critical.

For every component C of G' linked to edge xy of matching M_v we can say that x is vertex which is linked to vertex v and y is linked to component C. Conversely, if y is not linked to C then x is linked to C. From Claim 3 follows that y is linked to vertex v and we can label x as y and vice versa.

Claim 4. Let G' has at least two components. Then the following statement holds for every edge xy of matching M_v. If x is linked to $x' \in C$ where C is a component of G', then y is linked to $y' \in C$, $y' \neq x'$.

Proof of Claim 4. We prove the claim by contradiction. Suppose y is not linked with any vertex of C. Let C_1 be a component of G' other than C. Let $x_1y_1 \in M_v$ be an edge, with $xy \neq x_1y_1$ and let y_1', resp. v, be linked with y_1, resp. x_1. Since, G is 2-connected and from structure of G, C_1 is connected to at least two vertices of M_v. From Claim 2 follows that one of them is not vertex incident with edge xy, hence such edge x_1y_1 exists. Let M be a matching defined by $M = (M_v \setminus \{\{xy\}) \cup (\{x_1y_1\}\} \cup \{xx'\} \cup \{y_1y_1'\})$. Every maximal matching $M' \supseteq M$ leaves uncovered one vertex of C, one vertex of C_1 (both $C \setminus \{x'\}$ and $C_1 \setminus \{y_1'\}$ are odd and not linked to any other uncovered vertex) and y. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 5. Let x_1y_1, x_2y_2 be edges in M_v and C_1, C_2 two different components of G' such that x_1 is adjacent to vertex $x_1' \in C_1$ and x_2 is adjacent to $x_2' \in C_2$. Then there does not exist an edge between $\{x_1, y_1\}$ and $\{x_2, y_2\}$.

Proof of Claim 5. We prove the claim by contradiction. Let e be such an edge. Due to Claim 4 y_1 is adjacent to $y_1' \in C_1$ and y_2 is adjacent to $y_2' \in C_2$. WLOG let the desired edge e be x_1x_2. Consequently, every maximal matching M' which is a superset of matching M defined by $M = (M_v \setminus (\{x_1y_1\} \cup \{x_2y_2\})) \cup (\{x_1x_2\} \cup \{y_1y_1'\} \cup \{y_2y_2'\})$ leaves unmatched at least one vertex of C_1, one vertex of C_2, and vertex v. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 6. Let x_1y_1, x_2y_2, x_3y_3, x_4y_4 be edges in M_v and let C_1, C_2 be two different components of G' such that x_1 is adjacent to $x_1' \in C_1$, x_2 is adjacent to $x_2' \in C_2$. Suppose x_3y_3 is not linked to C_1 and x_4y_4 is not linked to C_2. If there is an edge between $\{x_1, y_1\}$ and $\{x_3, y_3\}$, then there is not an edge between $\{x_2, y_2\}$ and $\{x_4, y_4\}$.

Proof of Claim 6. We prove the claim by contradiction. Let e, f be such edges. Due to Claim 4 y_1 is adjacent to $y_1' \in C_1$ and y_2 is adjacent to $y_2' \in C_2$. From Claim 5 and the fact that x_3y_3 is not linked to C_1 but is linked by e to x_1y_1 follows that x_3y_3 is not linked to any component of G'. Analogously x_4y_4 is not linked to any component of G'. WLOG let e be x_1x_3 and f be x_2x_4. Consequently, every maximal matching M' which is superset of matching M defined by $M = M_v \setminus (\{x_1y_1\} \cup \{x_2y_2\} \cup \{x_3y_3\} \cup \{x_4y_4\}) \cup (\{x_1x_3\} \cup \{x_2x_4\} \cup \{y_1y_1'\} \cup \{y_2y_2'\})$ leaves unmatched at least one vertex of C_1, one vertex of C_2 and one of vertices v, y_3, y_4. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 7. Let x_1y_1, x_2y_2, x_3y_3 be edges in M_v and let C_1, C_2 be two different components of G' such that x_1 is adjacent to $x_1' \in C_1$, x_2 is adjacent to $x_2' \in C_2$. Suppose that x_3y_3 is not linked to C_1. If there is an edge between $\{x_1, y_1\}$ and $\{x_3\}$, then there is not an edge between $\{x_2, y_2\}$ and $\{y_3\}$.

Proof of Claim 7. We prove the claim by contradiction. Let e, f be such edges. Due to Claim 4 y_1 is adjacent to $y_1' \in C_1$ and y_2 is adjacent to $y_2' \in C_2$. From Claim 5 and the fact that x_3y_3 is not linked to C_1 but is linked by e to x_1y_1 follows that x_3y_3 is not linked to any component of G'. WLOG let e be edge x_1x_3 and f be edge y_2y_3. Consequently, every maximal matching M' which is superset of matching M defined by $M = M_v \setminus (\{x_1y_1\} \cup \{x_2y_2\} \cup \{x_3y_3\}) \cup (\{x_1x_3\} \cup \{y_2y_3\} \cup \{y_1y_1'\} \cup \{x_2x_2'\})$ leaves unmatched at least one vertex of C_1, one vertex of C_2 and vertex v. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 8. Let x_1y_1, x_2y_2, x_3y_3 be edges in M_v. Let C_1, C_2 be two different components of G' such that x_1 is adjacent to $x_1' \in C_1$, x_2 is adjacent to $x_2' \in C_2$. Suppose that x_3y_3 is not linked to C_1. If there is an edge between $\{x_1, y_1\}$ and $\{x_3\}$, then there is not an edge between $\{x_2, y_2\}$ and $\{x_3\}$.

Proof of Claim 8. We prove the claim by contradiction. Let e, f be such edges. Due to Claim 4 y_1 is adjacent to $y_1' \in C_1$ and y_2 is adjacent to $y_2' \in C_2$. From Claim 5 and the fact that x_3y_3 is not linked to C_1 but is linked by e to x_1y_1 follows that x_3y_3 is not linked to any component of G'. WLOG let edge e be x_1x_3 and f be edge y_2y_3 and x_2 be linked to vertex v (if x_2 is not linked to v, then y_2 is and we change x_2 for y_2 and x_2' for y_2' and vice versa). Consequently, every maximal matching M' which is superset of matching M defined by $M = M_v \setminus (\{x_1y_1\} \cup \{x_2y_2\} \cup \{x_3y_3\}) \cup (\{x_1x_3\} \cup \{y_2y_2'\} \cup \{y_1y_1'\} \cup \{x_2v\})$ leaves unmatched at least one vertex of C_1, one vertex of C_2, and vertex y_3. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 9. If G' has at least two components C_1 and C_2, then v is a cutvertex.

Proof of Claim 9. WLOG C_1 is linked to x_1y_1 and C_2 is linked to x_2y_2 where x_1y_1 and x_2y_2 are two different edges of M_v. Clearly, C_1 is not linked to C_2 since they are different components of G'. From claims Claim 4 and Claim 2 follows that C_1 cannot be linked to edge x_2y_2. Let x_2y_2 be linked to edge e of M_v. Then in spite of Claim 5 C_1 cannot be linked directly to e. From claims Claim 6, Claim 7, Claim 8 follows that x_1y_1 cannot be linked to e or to another edge f which is not linked to C_1 and therefore there could be path between f and e, x_2y_2, or C_2, that does not contains v. This is true for arbitrary x_1y_1 linked to C_1 and x_2y_2 linked to C_2 therefore every path from C_1 to C_2 goes through v and v is a cutvertex.

Using Claim 9 and the fact that G is 2-connected, it is easy to show that G' has only one component. From Claim 1 follows that this component is either K_{2n} or $K_{n,n}$. □

Lemma 3.4. *Let G be a graph of orientable genus g. If G is either K_{2n} or $K_{n,n}$ then $|V(G)| \leq 4 + 4\sqrt{g}$.*

Proof. If G is K_{2n} then $|V(G)| \leq (1/2)(7 + \sqrt{1 + 48q})$. If G is $K_{n,n}$ then $|V(G)| \leq 4 + 4\sqrt{g}$ (see Theorems 1.3.3 and 1.3.4 on page 9). For every $g \geq 0$ inequality $(7 + \sqrt{1 + 48q})/2 \leq 4 + 4\sqrt{g}$ holds, hence $|V(G)| \leq 4 + 4\sqrt{g}$. □

Lemma 3.5. *Let G be a 2-connected, factor-critical equimatchable graph of orientable genus g. Let $v \in V(G)$ be a vertex with minimal degree d in G. Then $V(G) \leq 5 + 2d + 4\sqrt{g}$.*

Proof. Let M_v be a minimal matching that isolates v. From Lemma 3.2 M_v covers at most $2d$ vertices. Let $G' = G \setminus (V(M_v) \cup \{v\})$. Due to Theorem 3.3 G' has at most one component that is either K_{2n} or $K_{n,n}$ and since Lemma 3.4 $|V(G')| \leq 4 + 4\sqrt{g}$. Hence G is a union of vertex v, matching M_v, and G',

$$|V(G)| \leq 1 + 2d + 4 + 4\sqrt{g} = 5 + 2d + 4\sqrt{g}.$$

□

Lemma 3.6. *If graph G with genus g has more than*

$$\frac{12(g-1)}{d-5}$$

vertices for some $d \geq 6$, then there exists vertex v with $deg(v) \leq d$.

Proof. We prove the lemma by contradiction. Let G be graph with $deg(w) \geq d+1$ for all $w \in V(G)$ and let $\Pi : G \to S$ be a 2-cell embedding in the surface of genus g. Let p denote the number of vertices of G, q the number of edges and r the number of faces of Π. Since $deg(w) \geq d+1$ we have $2q \geq (d+1) \cdot p$.

Since G is a simple graph, from Theorem 1.3.8 on page 10 $2q \geq 3r$.

From Euler-Pointcaré formula (see Theorem 1.3.7) follows

$$2 - 2g = p - q + r \leq \frac{2q}{d+1} - q + \frac{2q}{3} = \frac{q(5-d)}{3(d+1)}.$$

Since $d \geq 6$ and $2q \geq (d+1)p$ we have

$$\frac{q(5-d)}{3(d+1)} \leq \frac{p(d+1)}{2} \cdot \frac{5-d}{3(d+1)}$$

so

$$2 - 2g \leq \frac{p(5-d)}{6}$$

which contradicts the assumption of the lemma. □

Corollary 3.7. *Let G be a 2-connected, factor critical, equimatchable graph with genus g. If G has more than $\frac{12(g-1)}{d-5}$ vertices for some $d \geq 6$, then G has less than $5+2d+4\sqrt{g}$ vertices.*

Theorem 3.8. *Let G be a 2-connected, factor-critical equimatchable graph embeddable in surface of orientable genus g. Then:*
(a) If $g \leq 2$, then $|V(G)| \leq 17 + 4\sqrt{g}$.
(b) If $g \geq 3$, then $|V(G)| \leq 5 + 12\sqrt{g}$.
(c) If $g \geq 63$, then $|V(G)| \leq 5 + 8\sqrt{g}$.
(d) If $g \to \infty$, then $|V(G)| \leq 5 + 2(1+\sqrt{7})\sqrt{g}$.

Proof. Let g_0 denote genus of graph G. Since G is embeddable in surface of genus g, $g_0 \leq g$. *(a)* If $g \leq 2$, then the inequality $17+4\sqrt{g} > 12\,(g-1)$ holds. From Lemma 3.6 follows that graph G has either at most $12\,(g_0-1) \leq 12\,(g-1)$ vertices or a vertex of degree at most 6, and hence by Lemma 3.5 at most $17+4\sqrt{g_0} \leq 17+4\sqrt{g}$ vertices.

(b) If $g \geq 3$, then the inequality

$$5 + 12\sqrt{g} > \frac{12\,(g-1)}{4\sqrt{g}-5}$$

holds. From Lemma 3.6 follows that G has either at most $12\,(g_0-1)\,/\,\left(4\sqrt{g}-5\right) \leq 12\,(g-1)\,/\,\left(4\sqrt{g}-5\right)$ vertices or a vertex of degree at most $4\sqrt{g}$, and hence by Lemma 3.5 at most $5 + 2\cdot 4\sqrt{g} + 4\sqrt{g_0} \leq 5 + 12\sqrt{g}$ vertices.

(c) If $g \geq 63$, then the inequality

$$5 + 8\sqrt{g} > \frac{12\,(g-1)}{2\sqrt{g}-5}$$

holds. By following the proof of part *(b)* we get $|V(G)| \leq 5 + 8\sqrt{g}$.

(d) From identity

$$\lim_{g\to\infty} 2(1+\sqrt{7})\sqrt{g} = \lim_{g\to\infty} \frac{12\,(g-1)}{(\sqrt{7}-1)\sqrt{g}-5}$$

and from Lemmas 3.6 and 3.5 follows that if $g \to \infty$, then $|V(G)| \leq 5 + 2(1+\sqrt{7})\sqrt{g}$.

□

Lemma 3.9. *Let H_1 be isomorphic to a $K_{n,n}$ and H_2 be isomorphic to a $K_{m,m+1}$. Let u and v be vertices of different partitions of H_1. Let x and y be different vertices from* *the larger partition of H_2. Then the graph G defined by $G = H_1 \cup H_2 \cup (ux \cup vy)$ is factor-critical and equimatchable.*

Proof. First, we show that G is factor-critical. That is, for any vertex w the graph $G \setminus \{w\}$ has a perfect matching. We distinguish three cases.

Case 1: w is a vertex of H_1. WLOG w is in same partition as v. Clearly, there is a perfect matching M_1 of $H_1 \setminus \{u, w\}$ and a perfect matching M_2 of $C_2 \setminus \{x\}$. It follows that matching M defined by $M = M_1 \cup M_2 \cup \{ux\}$ is perfect matching of $G \setminus \{w\}$.

Case 2: w is vertex of A. Similarly, there is perfect matching M_1 of H_1 and a perfect matching M_2 of $H_2 \setminus \{w\}$. Therefore, matching M defined by $M = M_1 \cup M_2$ is perfect matching of $G \setminus \{w\}$.

Case 3: w is vertex of B. There is a perfect matching M_1 of $H_1 \setminus \{u, v\}$ and perfect matching M_2 of $H_2 \setminus \{x, y\}$. Therefore, matching M defined by $M = M_1 \cup M_2 \cup \{ux, vy\}$ is perfect matching of $G \setminus \{w\}$.

Now we show that G is equimatchable. Any maximum matching of the graph G covers $2(m + n)$ vertices. One of maximum matchings can be obtained as the union of maximum matchings of H_1 and H_2. Graphs $K_{n,n}$ and $K_{n+1,n}$ are equimatchable for any n. Therefore, graphs H_1, $H_1 \setminus \{u\}$, $H_1 \setminus \{u, v\}$, H_2, $H_2 \setminus \{x\}$, and $H_2 \setminus \{x, y\}$ are equimatchable. Every matching that does not contain neither edge ux, nor edge vy is the union of matchings of graphs H_1 and H_2, hence can be extended to a maximum matching. Similarly, every matching that contains edge ux (resp. vy) is the union of ux (resp. vy), a matching of $H_1 \setminus \{u\}$ (resp. $H_1 \setminus \{v\}$), and a matching of $H_2 \setminus \{x\}$ (resp. $H_2 \setminus \{y\}$), therefore can be extended to a maximum matching. Finally, every matching that contains ux and vy is the union of ux, vy, a matching of $H_1 \setminus \{u, v\}$, and a matching of $H_2 \setminus \{x, y\}$, therefore can be extended to a maximum matching. The fact that G is equimatchable follows. □

Lemma 3.10. *For any nonnegative integer g there exists a 2-connected factor-critical equimatchable graph embeddable in the surface of orientable genus g and at least $\lfloor\sqrt{8g}\rfloor + \lfloor\sqrt{8g+1}\rfloor + 6$ vertices.*

Proof. Let H_1 be a graph isomorphic to $K_{n,n}$ with $\lfloor\sqrt{8g}\rfloor + 4$ (resp. $\lfloor\sqrt{8g}\rfloor + 3$) vertices, if $\lfloor\sqrt{8g}\rfloor$ is even (resp. odd). Let H_2 be a graph isomorphic to $K_{m+1,m}$ with $\lfloor\sqrt{8g+1}\rfloor + 3$ (resp. $\lfloor\sqrt{8g+1}\rfloor + 4$) vertices, if $\lfloor\sqrt{8g+1}\rfloor$ is even (resp. odd). From

Theorem 1.3.2, Theorem 1.3.3, and Theorem 1.3.5 follows that H_1 and H_2 have 2-cell embeddings in the surface of orientable genus $g/2$. Let u and v be vertices of different partitions of H_1. Let Π be a minimal genus embedding of H_2. Every region f in Π has length at least four. Since, H_2 is bipartite, half of vertices of boundary walk of region f is from larger partition. Let x and y be such vertices of larger partition of H_2 that are in the same face of Π. Let G be graph defined by $G = H_1 \cup H_2 \cup (ux \cup vy)$. Clearly, graph $G' = H_1 \cup H_2 \cup ux$ has a 2-cell embedding in the surface of orientable genus g. Since edge uv is in $E(H_1)$, in every embedding of H_1 there exists a face f such that vertices u and v are in f. Also x and y are in the same face in Π. Therefore,there exists embedding of graph G' with v and y in same face. Adding edge vy to such embedding will not increase genus of graph. The fact that G has orientable genus g follows.

Clearly, G is 2-connected. Due to Lemma 3.9 graph G is factor-critical and equimatchable. Therefore, the statement from the lemma holds. $\square$

Theorem 3.11. *Let $f(g)$ be function that gives the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the surface of orientable genus g. Then:*
i) If $g \leq 2$, then $5\sqrt{g} + 6 \leq f(g) \leq 4\sqrt{g} + 17$.
ii) If $g \geq 3$, then $5\sqrt{g} + 6 \leq f(g) \leq 12\sqrt{g} + 5$.
iii) If $g \geq 63$, then $5\sqrt{g} + 6 \leq f(g) \leq 8\sqrt{g} + 5$.

Proof. If $g \geq 0$, then $5\sqrt{g} \leq \lfloor\sqrt{8g}\rfloor + \lfloor\sqrt{8g+1}\rfloor$. Therefore, from Lemma 3.10 follows $f(g) \geq 6 + 5\sqrt{g}$. Upper bounds in *i)*, *ii)*, *iii)* follows respectively from Theorem 3.8 part *(a)*, *(b)*, *(c)*.

$\square$

Lemma 3.12. *Let G be a 2-connected, factor-critical equimatchable graph embeddable in surface of orientable genus g. Let v be a vertex of G, and suppose M_v is a matching that isolates v. Let H_1 be subgraph of G defined by $H_1 = G \setminus (\{v\} \cup M_v)$. If there exists a vertex w that is not adjacent to any vertex of M_v, then subgraph H_2 of G defined by $H_2 = M_v \cup \{v\}$ is one of the following. K_{2n+1} (perhaps without one edge), $K_{n+1,n}$ (perhaps with one edge between edges of the $(n+1)$-stable set), $K_{n,n} \cup \{a\}$, or $K_{2n} \cup \{a\}$, where a is a new vertex adjacent to at least one other vertex of the graph. Moreover, in the last two cases, any embedding of subgraph H_1 in the surface is such that every point of H_1 lies in same face.*

Proof. Since G is 2-connected, there are at least two independent paths between H_2 and H_1. Therefore, there exist different vertices $c_1, c_2 \in H_1$ and $a_1, a_2 \in H_2$ such that there are edges a_1c_1 and a_2c_2. From Theorem 3.3 follows that H_1 is isomorphic either to K_{2n} or to $K_{n,n}$.

Case 1: H_1 is a K_{2n}. Let w be a vertex in H_1 that is not adjacent to any vertex of H_2. Clearly, w is adjacent to vertices c_1 and c_2. Let M be a perfect matching of H_1 such that the edge wc_1 is in M. Every maximal matching of G that is a superset of matching $(M \setminus \{wc_1\}) \cup a_1c_1$ necessarily leaves w uncovered. Since G is equimatchable and factor critical, $H_2 \setminus a_1$ has to be randomly matchable. Therefore, $H_2 \setminus a_1$ is $K_{m,m}$ or K_{2m} for some m. Analogously, also $H_2 \setminus a_2$ is $K_{m,m}$ or K_{2m} for some m. Therefore, H_2 could be only K_{2m+1}, possibly without the edge a_1a_2, or $K_{m,m+1}$, possibly without the edge a_1a_2, where a_1 and a_2 are both in $(m+1)$-stable set.

Case 2: H_1 is $K_{n,n}$. Clearly, we can choose vertices c_1 and c_2 such that at least one of them (WLOG c_1) is adjacent to a vertex that is not adjacent to any vertex of H_2.

Case 2.1: both c_1 and c_2 have at least one neighbour from H_1 that is not adjacent to any vertex of H_2. The proof is almost identical with the proof of *Case* 1.

Case 2.2: Every neighbour of c_2 from H_1 is adjacent to some vertex of H_2. Let w be a vertex that is adjacent to c_1 without neighbours from H_2 and suppose M is a perfect matching of H_1 containing edge wc_1. Similarly as in *Case 1* we get that $H_2 \setminus a_1$ is $K_{m,m}$ or K_{2m} for some m. Since every neighbour of c_2 that is in H_1 is adjacent to some vertex of H_2, every vertex of the partition V of H_1 which does not contain c_2 is adjacent to at least one vertex of H_2. Therefore, V has to be whole on outer face f of H_1 in every embedding of G. None of vertices in V are adjacent and every face in embedding of 2-connected graph H_2 is cycle. Therefore, there have to be $|V|$ next vertices in f. Clearly, embedding of H_1 is such that every point of H_1 lies in same face. □

Corollary 3.13. *Let G be graph with the maximum amount of vertices such that:*
1) G is a 2-connected, factor-critical equimatchable graph of orientable genus g.
2) There are vertices $v, w \in V(G)$ and a matching M_v isolating v such that w is not vertex of M_v and is not adjacent to any vertex of M_v.
Then there exists k, with $4 \leq k \leq 9$ such that $|V(G)| = 4\sqrt{2g} + k$.

Proof. From Lemma 3.10 there exists such a graph with $\lfloor\sqrt{8g}\rfloor + \lfloor\sqrt{8g+1}\rfloor + 6 \geq$

$4\sqrt{2g}+4$ vertices.

Let v and M_v be a vertex and its isolating matching as defined in *2)*. Let $H_2 = M_v \cup \{v\}$ and $H_1 = G \setminus H_2$. By Theorem 3.3 is H_1 either K_{2n} or $K_{n,n}$. From Lemma 3.12 follows that H_2 is one of following: K_{2n+1} (perhaps without one edge), $K_{n+1,n}$ (perhaps with one edge between edges of the $(n+1)$-stable set), $K_{n,n} \cup \{a\}$, or $K_{2n} \cup \{a\}$, where a is a new vertex adjacent to at least one other vertex of the graph. Clearly, if H_1 has genus g_1 and H_2 has genus g_2, then G has genus at least $g_1 + g_2$. From Theorem 1.3.3 and Theorem 1.3.4 on page 9 follows that if H_1 has genus g_1, then it has at most $4 + \sqrt{g_1}$ vertices. It is easy to see that $\gamma(K_{2n+1}) \geq \gamma(K_{2n} \cup \{a\})$ and $\gamma(K_{n,n+1}) \geq \gamma(K_{n,n} \cup \{a\})$, and from Theorem 1.3.3 and Theorem 1.3.4 follows that $\gamma(K_{2n}) \geq \gamma(K_{n,n})$. Therefore, if H_2 has genus g_2, then H_2 has at most $5+\sqrt{g_2}$ vertices. Hence, $|V(G)| \leq 9 + 4(\sqrt{g_1} + \sqrt{g_2})$, with $g_1 + g_2 = g$. From Jensen's inequality (see [Jen06]) follows $|V(G)| \leq 9 + 4\sqrt{2g}$. $\square$

Conclusion

This thesis deals with the maximum size of equimatchable graphs embeddable in a fixed surface, with the focus being on factor-critical equimatchable graphs. Thesis resulted in the following theorem.

Theorem. *Let $f(g)$ be function that gives the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the surface of orientable genus g. Then:*
i) If $g \leq 2$, then $5\sqrt{g} + 6 \leq f(g) \leq 4\sqrt{g} + 17$.
ii) If $g \geq 3$, then $5\sqrt{g} + 6 \leq f(g) \leq 12\sqrt{g} + 5$.
iii) If $g \geq 63$, then $5\sqrt{g} + 6 \leq f(g) \leq 8\sqrt{g} + 5$.

Our proof is based on the following Theorem describing the graph obtained by a removal of a minimal isolating matching from an equimatchable graph.

Theorem. *Let G be a 2-connected, factor-critical equimatchable graph. Let $v \in V(G)$ be a vertex of G and M_v minimal matching that isolates v. Let $G' = G \setminus (V(M_v) \cup \{v\})$. Then G' is isomorphic with K_{2n} or a $K_{n,n}$ for some nonnegative integer n.*

We extend the results from [KPS03] by showing that when we allow graphs that are not 2-connected, there are infinitely many equimatchable planar connected graphs, both factor critical and bipartite. In addition to these results, we provide novel structural description of factor-critical equimatchable graphs and graphs embeddable in a fixed surface.

Among the most prominent open problems left in the area are the following.
1) What is the exact upper bound on the number of vertices of a 2-connected, factor-critical equimatchable graph embeddable in the surface of a fixed genus g?
2) Are there only finitely many 2-connected bipartite equimatchable graphs embeddable in the surface of genus g with representativity at least 3?
3) Are there only finitely many 3-connected bipartite equimatchable graphs embeddable in the surface of genus g?

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