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FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY

ALTERNATING WEIGHTED AUTOMATA OVER COMMUTATIVE SEMIRINGS

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Filip Mišún

COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Alternating Weighted Automata over Commutative Semirings

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Department: Department of Computer Science

Supervisor: RNDr. Peter Kostolányi

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Filip Mišún

60680610

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta: Filip Mišún
Študijný program: informatika (Jednoodborové štúdium, bakalársky I. st., denná forma)
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Názov: Alternating Weighted Automata over Commutative Semirings
Alternujúce automaty s váhami nad komutatívnym polokruhom

Cieľ: Definovať alternujúce automaty s váhami nad ľubovoľným komutatívnym polokruhom ako zovšeobecnenie booleovských alternujúcich automatov, v ktorom sú existenčné a univerzálne stavy nahradené stavmi zodpovedajúcimi súčtom a súčinom v danom polokruhu. Začať so skúmaním vlastností takýchto automatov a nimi realizovaných formálnych mocninových radov.

Vedúci: RNDr. Peter Kostolányi
Katedra: FMFI.KI - Katedra informatiky
Vedúci katedry: prof. RNDr. Martin Škoviera, PhD.

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Dátum schválenia: 14.10.2016

doc. RNDr. Daniel Olejár, PhD.
garant študijného programu

....
študent vedúci práce

60680610

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname:	Filip Mišún
Study programme:	Computer Science (Single degree study, bachelor I. deg., full time form)
Field of Study:	Computer Science, Informatics
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Title: Alternating Weighted Automata over Commutative Semirings

Aim: To define alternating weighted automata over an arbitrary commutative semiring by generalisation of Boolean alternating automata, in which existential and universal states are replaced by states corresponding to sums and products in the given semiring. To initiate the study of properties of such automata and of formal power series, which they realise.

Supervisor:	RNDr. Peter Kostolányi
Department:	FMFI.KI - Department of Computer Science
Head of department:	prof. RNDr. Martin Škoviera, PhD.
Assigned:	28.09.2016
Approved:	14.10.2016

doc. RNDr. Daniel Olejár, PhD.
Guarantor of Study Programme

.... Student

.... Supervisor

Abstrakt

V práci definujeme a skúmame nové rozšírenie alternujúcich konečných automatov, v ktorom má každý prechod priradenú váhu z nejakého komutatívneho polokruhu, disjunkcie sú nahradené súčtami a konjunkcie sú nahradené súčinmi. Takto definované automaty nazývame alternujúcimi automatmi s váhami. Skúmame triedu formálnych mocninových radov realizovaných alternujúcimi automatmi s váhami a rôzne spôsoby, ktorými možno túto triedu charakterizovať. Hlavným výsledkom dokázaným v práci je charakterizácia triedy komutatívnych polokruhov, pre ktoré sú automaty s váhami a alternujúce automaty s váhami rovnako silné. Skúmame uzáverové vlastnosti tried formálnych mocninových radov realizovaných alternujúcimi automatmi s váhami.

Kľúčové slová: alternujúci automat s váhami, alternácia, formálny mocninový rad, komutatívny polokruh

Abstract

We define and begin the study of alternating weighted automata, a new extension of alternating finite automata, in which transitions carry weights given by elements of some commutative semiring. In this extension, disjunctions are replaced by sums and conjunctions are replaced by products of the semiring in consideration. We study various different ways, in which one can characterize formal power series realized by alternating weighted automata. We prove a characterization of the class of commutative semirings, for which weighted automata and alternating weighted automata are equally powerful. We also examine closure properties of the classes of formal power series realized by alternating weighted automata under several standard operations.

Keywords: alternating weighted automaton, alternation, formal power series, commutative semiring

Contents

Introduction

The goal of this thesis is to define and begin the study of alternating weighted automata, a new extension of alternating finite automata [2], in which transitions carry weights given by elements of some commutative semiring. In our extension, disjunctions are replaced by sums, while conjunctions are replaced by products of the semiring in consideration. Alternating weighted automata realize formal power series and one can also view them as an extension of weighted automata [4].

This thesis continues in the study of alternation in weighted automata initiated by Chatterjee, Doyen and Henzinger [3]. In their article, they introduced and studied (among some other particular settings) weighted automata over infinite words and over the tropical semiring with states performing both "min" and "max" operations. The usefulness of this model was justified by its possible applications in formal verification of reactive systems. The study of alternation in weighted automata over the tropical semiring was later continued by Almagor and Kupferman [1], who have focused on automata over *finite* words. Besides automata with "min-max" alternation, they have also studied "min-sum" alternating automata, arguing that this type of alternation is useful for the purposes of formal verification as well. Note that in this setting, addition in the tropical semiring alternates with multiplication in the same semiring.

Alternating weighted automata as we define them in this thesis are a generalization of weighted automata with min-sum alternation to an arbitrary commutative semiring; in our model, addition in some commutative semiring alternates with multiplication in the same semiring. This makes weighted automata with min-sum alternation studied by Almagor and Kupferman [1] just a special case of our object of study. The goal of this thesis is to study alternating weighted automata in this more general setting from a theoretical point of view. Our definition of alternating weighted automata will not incorporate min-max alternating automata of Chatterjee, Doyen and Henzinger [3].

We shall give two alternative definitions of alternating weighted automata. To be more precise, we shall introduce two different, but equivalent models with these definitions. In one of these models, there are two types of states: "sum" states that can only perform addition and "product" states that can only perform multiplication. In the other model, each state can combine additive and multiplicative operation. Once we state these two definitions, we shall prove that our two models are equally powerful. As a next step, we shall prove some basic results on alternating weighted automata. We shall present a construction that can be used to eliminate ε-labelled transitions in alternating weighted automata. Subsequently, we shall introduce systems of H-polynomial equations that provide a different characterization of formal power series realized by alternating weighted automata.

After these basic considerations, we shall focus our attention on the main result of this thesis: the comparison of power of weighted automata and alternating weighted automata. Weighted automata are just a special case of alternating weighted automata and for this reason, alternating weighted automata are at least as powerful as weighted automata. Almagor and Kupferman showed that alternating weighted automata over the tropical semiring are strictly more powerful than weighted automata over the same semiring [1]. On the other hand, it is a well known fact that every alternating finite automaton (without weights) realizes a regular language [2]. This implies that weighted automata over the Boolean semiring and alternating weighted automata over the Boolean semiring are equally powerful. We conclude that commutative semirings can be divided into two nonempty classes: the class of commutative semirings, for which alternating weighted automata and weighted automata are equally powerful and the class of commutative semirings, for which alternating weighted automata are strictly more powerful than weighted automata. In the most significant result of this thesis, we shall give a characterization of these two classes of commutative semirings.

Finally, we shall examine some standard closure properties of classes of formal power series realized by alternating weighted automata. For each commutative semiring S, one might examine the class of formal power series realized by alternating weighted automata over S. We shall prove that this class is closed under sum and Hadamard product for every commutative semiring S. On the other hand, we shall prove that these classes are not in general closed under Cauchy product and under reversal.

Chapter 1

Preliminaries

The aim of this thesis is to define and study alternating weighted automata, a new model that generalizes two well known extensions of finite automata: alternating finite automata and weighted automata. One of the goals of this chapter is to review the definitions and some of the basic properties of these two models. We shall also give the definition of semirings and explain some other basic notions from semiring theory that we shall need at some point in this thesis.

1.1 Semirings and Polynomials

Definition 1.1.1. A *semiring* is a tuple $(S, +, \cdot, 0, 1)$, where S is a set, 0 and 1 are elements of S, and $+, \cdot$ are binary operations on S such that

- $(S, +, 0)$ is a commutative monoid, i.e., $a + (b + c) = (a + b) + c$, $a + b = b + a$, and $a + 0 = 0 + a = a$ holds for every a, b, c in S;
- $(S, \cdot, 1)$ is a monoid, i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and $a \cdot 1 = 1 \cdot a = a$ holds for every a, b, c in S;
- multiplication distributes over addition, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ holds for every a, b, c in S;
- $a \cdot 0 = 0 \cdot a = 0$ holds for every a in S.

Definition 1.1.2. A semiring $(S, +, \cdot, 0, 1)$ is *commutative* if the multiplication is commutative, i.e., if $a \cdot b = b \cdot a$ holds for every a, b in S.

We shall be primarily interested in commutative semirings. Some of the more important commutative semirings are mentioned in the following list:

- The set $\mathbb{B} = \{0, 1\}$, together with the logical disjunction as addition and the logical conjunction as multiplication forms the *Boolean semiring* $(\mathbb{B}, \vee, \wedge, 0, 1)$.
- The set of nonnegative real numbers $\mathbb{R}_+$ with the standard operations of sum and product constitutes a semiring $(\mathbb{R}_+, +, \cdot, 0, 1)$.
- The set $\mathbb{R} \cup \{\infty\}$ of real numbers with positive infinity, together with minimum as addition and the standard addition of real numbers as multiplication forms a *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.
- The set $\mathbb{R} \cup \{-\infty\}$ of real numbers with negative infinity, together with maximum as addition and the standard addition of real numbers as multiplication constitutes an *arctic semiring* $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$.
- The powerset $\mathscr{P}(U)$ of an arbitrary set U with union as addition and intersection as multiplication forms the semiring $(\mathscr{P}(U), \cup, \cap, \emptyset, U)$.

- The set $2^{\{a\}^*}$ of languages over a singleton alphabet $\{a\}$, together with union as addition and concatenation as multiplication forms the semiring $(2^{\{a\}^*}, \cup, \cdot, \emptyset, \{\varepsilon\})$.

A *subsemiring* of a semiring $(S, +, \cdot, 0, 1)$ is a subset T of S that contains 0, 1 and is closed under addition and multiplication. If T is a subsemiring of S, then T forms a semiring together with the operations $+$ and $\cdot$ restricted to T. One can easily show that if $\mathcal{U}$ is a collection of subsemirings of S, then $\bigcap_{T \in \mathcal{U}} T$ is a subsemiring of S as well. If X is a subset of S and $\mathcal{U}$ is the collection of all subsemirings of S that contain X, we say that $\bigcap_{T \in \mathcal{U}} T$ is the *subsemiring generated by* X. The subsemiring generated by X is the smallest subsemiring of S (with respect to inclusion) that contains X. We say that a semiring S is *finitely generated* if it is generated by some finite subset of S.

Let a be an element of a semiring S. For every nonnegative integer n, we define $na := \sum_{i=1}^{n} a$ and $a^n := \prod_{i=1}^{n} a$. In particular, $0a = 0$ and $a^0 = 1$. The use of the symbols 0 and 1 to denote both the semiring elements and the integers may sometimes be confusing. For this reason, we sometimes prefer to write 0_S instead of 0 and 1_S instead of 1 to denote the zero and the unity element of S. More generally, we shall use the notation n_S to denote the element $n1_S$ for every nonnegative integer n.

We say that an element a of a semiring S has *finite additive order* if the set $\{na \mid n \in \mathbb{N}\}$ is finite. Otherwise, we say that a has *infinite additive order*. Similarly, we say that a has *finite multiplicative order* if the set $\{a^n \mid n \in \mathbb{N}\}$ is finite, and we say that a has *infinite multiplicative order* otherwise. The reader can easily check that a in S has finite additive order (or finite multiplicative order) iff there exist two distinct nonnegative integers n, m such that $na = ma$ (or $a^n = a^m$).

For every commutative semiring S, let $S[x_1, x_2, \ldots, x_n]$ denote the set of all polynomials in indeterminates $x_1, x_2, \ldots, x_n$ with coefficients in S. This set, together with the operations of addition and multiplication of polynomials derived from the operations of the semiring S in the usual way, constitutes a commutative semiring. If m in $S[x_1, \ldots, x_n]$ is such that

$$m = c x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$$

for some c in S and nonnegative integers $k_1, \ldots, k_n$, we call m a *monomial*. For each such monomial m, the coefficient c of m shall be denoted by $coef(m)$ and the exponent k_i of the indeterminate x_i shall be denoted by $exp(m, i)$ for $i = 1, \ldots, n$. If $k_1 = k_2 = \ldots = k_n = 0$, we say that m is a *constant*. The whole semiring $S[x_1, \ldots, x_n]$ is generated by its monomials. If P in $S[x_1, \ldots, x_n]$ can be written as a sum of nonconstant monomials,[1] we say that P has *zero constant term*. The subset of $S[x_1, \ldots, x_n]$ that consists of all polynomials with zero constant term is denoted by $S[x_1, x_2, \ldots, x_n]_{\mathrm{const}=0}$. Note that although this subset is closed under addition and multiplication, it is not a subsemiring of $S[x_1, \ldots, x_n]$, since it does not contain the unity of $S[x_1, \ldots, x_n]$.

For every commutative semiring S, we write $S(x_1, x_2, \ldots, x_n)$ to denote the set of all polynomial functions in indeterminates $x_1, x_2, \ldots, x_n$ with coefficients in S. This set, together with the operations of addition and multiplication of polynomial functions derived from the operations of the semiring S in the usual way, constitutes a commutative semiring.

1.2 Alternating Finite Automata

Alternating finite automata are an extension of finite automata that combines nondeterminism and parallelism. They were first formally defined and studied by Chandra, Kozen and Stockmeyer [2]. In one of the possible definitions, the states of an alternating finite automaton $\mathcal{A}$ are of two types: "existential" states and "universal" states. If a run in $\mathcal{A}$ is in some state q, there might be several transitions, which the run might follow. If q is an existential state, one of the transitions is chosen and the run follows this chosen transition. If the state q is universal, the run splits into multiple parallel branches and each transition is followed by one of these parallel branches.

[1] This is in particular true if $P = 0$.

In this thesis, we shall adopt a slightly more general definition of alternating finite automata [2], in which states are not strictly labelled as existential and universal, but nondeterminism and parallelism might be combined in each state instead. These combinations are expressed by *positive Boolean formulae without constants.* A positive Boolean formula without constants is a Boolean formula that contains only symbols for conjunction, disjunction, and variables (as an exception, the zero Boolean constant is also a positive Boolean formula without constants). We shall write $\mathbb{B}[x_1, x_2, \ldots, x_n]_{\mathrm{const}=0}$ to denote the set of all positive Boolean formulae without constants in variables $x_1, x_2, \ldots, x_n$.[2]

Inspired by the theory of formal power series, we shall also use the following notation. If L is a language and w is a word in L, let (L, w) be 1. If w is not in L, let (L, w) be 0. We are now prepared to give the definition of alternating finite automata.

Definition 1.2.1. A (*Boolean*) *alternating finite automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \psi, \phi_0, \tau)$, where Q is a finite set of states with $n := |Q|$; Σ is an alphabet; $\psi : (Q \times \Sigma) \to \mathbb{B}[x_1, \ldots, x_n]_{\mathrm{const}=0}$ is a formula assigning function; ϕ_0 in $\mathbb{B}[x_1, \ldots, x_n]_{\mathrm{const}=0}$ is an initial formula; $\tau \subseteq Q$ is a set of terminal states.

We shall always assume that the states of each alternating finite automaton are somehow linearly ordered. Although this linear ordering is not part of the definition above, it is nevertheless important, as we shall see in the following definition. Moreover, whenever we refer to "the i-th state", where i is a positive integer, we mean the i-th state with respect to the linear ordering of the states of the automaton in consideration.

Definition 1.2.2. Let $\mathcal{A} = (Q, \Sigma, \psi, \phi_0, \tau)$ be an alternating finite automaton, let $n = |Q|$. For each p in Q, we define a language $|\mathcal{A}|_p$ (also denoted by $|\mathcal{A}|_i$ if p is the i-th state of $\mathcal{A}$) over Σ as follows:

1. The language $|\mathcal{A}|_p$ contains ε iff p is in τ.
2. For each a in Σ and w in Σ^*, the language $|\mathcal{A}|_p$ contains aw iff

$$\psi[p, a]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big) = 1.$$

The *behaviour* of $\mathcal{A}$ is a language $|\mathcal{A}|$ over Σ defined as follows: for each w in Σ^*, the language $|\mathcal{A}|$ contains w iff

$$\phi_0\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big) = 1.$$

Let us now discuss the relationship between alternating finite automata and nondeterministic finite automata. We shall say that an alternating finite automaton $\mathcal{A} = (Q, \Sigma, \psi, \phi_0, \tau)$ is "disjunction-only" if the formula ϕ_0 contains no conjunction and neither does the formula $\psi[p, a]$ for each p in Q and a in Σ. Intuitively, a disjunction-only alternating finite automaton uses no parallelism. Indeed, it is quite easy to see that every nondeterministic finite automaton can be viewed as a disjunction-only alternating finite automaton and vice versa. Hence, every regular language is accepted by some alternating finite automaton. The less obvious fact is that the converse holds as well, i.e., every language accepted by an alternating finite automaton is regular. This is already a well known fact [2], but it will also follow from a more general statement on alternating weighted automata that we shall prove in Chapter 3.

1.3 Weighted Automata and Formal Power Series

Weighted automata, first introduced by Schützenberger [6], are an extension of nondeterministic finite automata, in which transitions carry weights given by elements of some semiring. If a run in a

[2] There is an obvious correspondence between positive Boolean formulae without constants and polynomials over the Boolean semiring with zero constant term. For this reason, we write $\mathbb{B}[x_1, \ldots, x_n]_{\mathrm{const}=0}$ to denote sets of both the former and the latter.

weighted automaton is in some state q and there are more transitions, which the run might follow, only one of these transitions is always chosen. Weights of the transitions that the run chooses to follow determine the weight of the run; this weight is obtained as a product over weights of all transitions that the run passes. A weighted automaton is nondeterministic in general and hence, multiple runs might be possible on the same word w. All these runs determine the weight of the word w; it is obtained as a sum over weights of all possible runs on w.

The behaviour of a weighted automaton is not a language; instead, it is a map that assigns a weight – a semiring element – to each word over its alphabet. Such maps are called formal power series [4].

Definition 1.3.1. A *formal power series* over a semiring S and over an alphabet Σ is a map from Σ^* to S.

If r is a formal power series over S and Σ and w is in Σ^*, then the value $r(w)$ is usually denoted by (r, w) and we call it the *coefficient of w in r*. The formal power series r itself is written as

$$r = \sum_{w \in \Sigma^*} (r, w)w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle\Sigma^*\rangle\rangle$.

We shall now define some operations on the set $S\langle\langle\Sigma^*\rangle\rangle$. For every r_1 and r_2 in $S\langle\langle\Sigma^*\rangle\rangle$, the *sum* of r_1 and r_2 is a formal power series $r_1 + r_2$ in $S\langle\langle\Sigma^*\rangle\rangle$ such that $(r_1 + r_2, w) = (r_1, w) + (r_2, w)$ for every w in Σ^*. The *Cauchy product* of r_1 and r_2 is a formal power series $r_1 \cdot r_2$ in $S\langle\langle\Sigma^*\rangle\rangle$ such that

$$(r_1 \cdot r_2, w) = \sum_{\substack{v_1, v_2 \in \Sigma^* \\ v_1 v_2 = w}} (r_1, v_1)(r_2, v_2)$$

for every w in Σ^*. By $r_1 \odot r_2$ we denote the formal power series in $S\langle\langle\Sigma^*\rangle\rangle$ satisfying $(r_1 \odot r_2, w) = (r_1, w)(r_2, w)$ for every w in Σ^*. We say that $r_1 \odot r_2$ is the *Hadamard product* of r_1 and r_2. The *n-th power* r^n of a formal power series r is defined inductively by

$$\begin{aligned} r^0 &= 1\varepsilon, \\ r^n &= r^{n-1} \cdot r, \qquad n \geq 1. \end{aligned}$$

Similarly, for every nonnegative integer n, we define a formal power series $r^{\odot\, n}$ inductively by

$$\begin{aligned} r^{\odot\, 0} &= \sum_{w \in \Sigma^*} 1w, \\ r^{\odot\, n} &= r^{\odot n-1} \odot r, \qquad n \geq 1. \end{aligned}$$

For every finite sequence $r_1, r_2, \ldots, r_n$ of formal power series in $S\langle\langle\Sigma^*\rangle\rangle$, we define $\sum_{i=1}^n r_i$, $\prod_{i=1}^n r_i$, and $\bigodot_{i=1}^n r_i$ to denote the sum, the Cauchy product, and the Hadamard product taken over all formal power series in the sequence $r_1, r_2, \ldots, r_n$.

Let us now define weighted automata.

Definition 1.3.2. Let S be a semiring. A *weighted automaton* over S is a tuple $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$, where Q is a nonempty finite set of states; Σ is an alphabet; T is a finite set of transitions, with which we associate maps *init*, *ter* : $T \to Q$ and $\sigma : T \to \Sigma \cup \{\varepsilon\}$; $\nu : T \to S$ is a transition weighting function; $\iota : Q \to S$ is an initial weighting function; $\tau : Q \to S$ is an terminal weighting function.

A weighted automaton can be viewed as a directed multigraph with labelled edges, where Q is the set of vertices and T is the set of edges. The maps *init* and *ter* assign initial and terminal vertex to each edge, while σ assigns labels.[3]

[3]Notice that the definition allows parallel edges (transitions) with the same label in the graph. This makes our definition rather unusual in comparison with definitions of majority of the authors [4]. Soon, we shall see that this nuance is inconsequential, meaning that it cannot change the expressive power of weighted automata. However, parallel transitions with the same label will play a significant role in our definition of *alternating* weighted automata and so we chose to allow them also in our definition of (nonalternating) weighted automata, as we want the definitions of these two models to be as similar as possible.

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be a weighted automaton, p be in Q and a be in $\Sigma \cup \{\varepsilon\}$. We shall write $T_{\mathcal{A}}(p)$ to denote the set of all transitions t in T such that $init(t) = p$, while $T_{\mathcal{A}}(p, a)$ shall be used to denote the set of all transitions t in T satisfying $init(t) = p$ and $\sigma(t) = a$. We shall write $T_{\mathcal{A}}^{\varepsilon}$ to denote the set of all ε-labelled transitions of $\mathcal{A}$. If clear from the context, we shall often omit the subscript denoting the automaton in consideration.

It is problematic to define the behaviour of a weighted automaton if its graph contains cycles of ε-labelled transitions. One of the usual solutions to this problem is to define the behaviour only for so-called *cycle-free* automata [5]. Under some conditions, a cycle-free weighted automaton might still contain cycles of ε-labelled transitions. It does not seem to be straightforward to make an analogous definition of cycle-free *alternating* weighted automata in such a manner that the definition does not become awkward. Therefore, our solution is to simply consider only those alternating weighted automata that contain no cycles of ε-labelled transitions. We shall follow the same approach for "ordinary" weighted automata as well, since we want to view them as a special case of alternating weighted automata.

Definition 1.3.3. Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be a weighted automaton. We say that $\mathcal{A}$ is *without ε-cycles* if the directed graph with vertex set Q and edge set T^{ε} contains no cycle.

It can be shown that every cycle-free weighted automaton is equivalent to a weighted automaton that contains no ε-labelled transitions at all [5]. This fact justifies our choice to consider only weighted automata without ε-cycles. From now on, whenever we refer to a weighed automaton, we shall always mean a weighted automaton without ε-cycles.

We can now finally define the behaviour of weighted automata.

Definition 1.3.4. Let S be a semiring and $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be a weighted automaton over S without ε-cycles. For each p in Q, let us define a formal power series $|\mathcal{A}|_p$ in $S\langle\langle \Sigma^* \rangle\rangle$ as follows:

1. The coefficient of ε in $|\mathcal{A}|_p$ is defined by

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, \varepsilon).$$

2. If a is in Σ and w is in Σ^*, then

$$(|\mathcal{A}|_p, aw) = \sum_{t \in T(p,a)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, w) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, aw).$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle \Sigma^* \rangle\rangle$ defined for all w in Σ^* by

$$(|\mathcal{A}|, w) := \sum_{p \in Q} \iota(p) \cdot (|\mathcal{A}|_p, w).$$

The definition of behaviour of weighted automata given above might seem to be quite different from the definition that we have anticipated at the beginning of this section. This difference is only superficial, though. If $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ is a weighted automaton and w is a word over Σ, we can define a run on w as a sequence $q_1, t_1, q_2, t_2, \ldots, t_{n-1}, q_n$, such that q_i is in Q for $i = 1, \ldots, n$, such that t_i is a transition with $init(t_i) = q_i$ and $ter(t_i) = q_{i+1}$ for $i = 1, \ldots, n-1$, and $w = \sigma(t_1)\sigma(t_2)\ldots\sigma(t_{n-1})$. For each such run, we define its weight to be $\iota(q_1) \cdot \left(\prod_{i=1}^{n-1} \nu(t_i)\right) \cdot \tau(q_n)$. The reader can check that for every word w over Σ, the coefficient of w in $|\mathcal{A}|$ is equal to the sum of weights of all runs on w.

We say that a formal power series r in $S\langle\langle \Sigma^* \rangle\rangle$ is *rational* over S if it is realized by some weighted automaton over S. The reader might have noticed that a weighted automaton over the Boolean semiring $\mathbb{B}$ can actually be viewed as a nondeterministic finite automaton and vice versa. Indeed, if r is a formal power series in $\mathbb{B}\langle\langle \Sigma^* \rangle\rangle$ and L is a language that consists of all w in Σ^* for which $(r, w) = 1$, then r is rational over $\mathbb{B}$ iff L is regular.

The theory of weighted automata constitutes a rich field in automata theory. In this section, we have only stated some definitions that are necessary for the purposes of this thesis. For a more comprehensive source of information, we refer the reader to the Handbook of Weighted Automata [4].

Chapter 2

Definitions and Basic Results

In this chapter, we introduce and begin the study of alternating weighted automata, a new model that extends both alternating finite automata (without weights) and weighted automata. In the previous chapter, we have described weighted automata as nondeterministic finite automata that assign a weight to each word over their alphabet. We saw that a nondeterministic finite automaton can be viewed as a weighted automaton over the Boolean semiring. In the same way, we shall define alternating weighted automata as a weight-assigning extension of alternating finite automata (without weights). Every state of an alternating finite automaton (without weights) is equipped with a set of positive Boolean formulae. In our generalization, we replace these formulae with polynomials over some commutative semiring; sums in such polynomials take the role of disjunctions, while products take the role of conjunctions. We shall see that an alternating finite automaton (without weights) can be viewed as an alternating weighted automaton over the Boolean semiring; this will be guaranteed by the obvious correspondence between positive Boolean formulae and polynomials over the Boolean semiring.

The term "alternating weighted automaton" was first used by Chatterjee, Doyen, and Henzinger [3]. Most importantly, they introduced weighted finite automata over infinite words and over the tropical semiring with states performing both "min" and "max" operations. Later, Almagor and Kupferman [1] studied "min-max" and "min-sum" alternation, this time in automata over finite words. With some changes in the definition of the min-sum model presented by Almagor and Kupferman, one can define alternating weighted automata with states performing operations of an arbitrary commutative semiring. Our goal is to define and study alternating weighted automata in this more general setting, the one defined by Almagor and Kupferman being just a special case of our object of study. On the other hand, our more general definition will not incorporate min-max alternating automata of Chatterjee, Doyen and Henzinger [3].

In this chapter, we shall first give a definition of alternating weighted automata as described above. Next, we shall introduce yet another new model – *two-mode* alternating weighted automata. The definition of two-mode alternating weighted automata will be very similar to the definition of (nonalternating) weighted automata; we shall simply introduce a new type of states. We shall prove that alternating weighted automata and two-mode alternating weighted automata are equally powerful. Therefore, the definition of two-mode alternating weighted automata can be viewed as an alternative definition of alternating weighted automata. Finally, we shall introduce what we shall call systems of H-polynomial equations. This new notion will allow us to give a different characterization of the formal power series realized by alternating weighted automata, which goes in the same lines as the well-known characterization of rational series in terms of linear systems [5].

2.1 Examples

Before we give a formal definition of alternating weighted automata, let us demonstrate their abilities on several examples. The following examples will hopefully provide the reader with a good enough intuition about the way alternating weighted automata operate.

As we have already anticipated, we shall give two alternative definitions of alternating weighted automata. To be more precise, these two definitions introduce two different models that are nevertheless equivalent. The first definition is given in the next section and it introduces automata that we shall simply call "alternating weighted automata". The definition of the second model is given in Section 2.4; instances of this model shall be called "two-mode alternating weighted automata". We chose to first give the definition of the former model, because it is simpler and also more general. On the other hand, the advantage of two-mode alternating weighted automata is that they are probably more intuitive and can be easily depicted by diagrams. For this reason, all examples that we give in this section describe constructions of two-mode alternating weighted automata.

Example 2.1.1. In Figure 2.1, we depict a two-mode alternating weighted automaton over the alphabet $\Sigma_1 = \{a, b\}$ and over the semiring of natural numbers with standard operations of addition and multiplication. Let us first explain the diagram in Figure 2.1. States of the automaton $\mathcal{A}_1$ are depicted by circles labelled either with "+" or with "×". If there is an arrow between two states of the automaton $\mathcal{A}_1$, this means that there is a transition between these two states. Every such arrow is labelled with a symbol from Σ_1; this symbol represents the label of the transition. The symbol of a transition might be preceded by a coefficient from the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ and it represents the weight of the transition. If the symbol of a transition is not preceded by a coefficient, then the weight of the transition is 1 (the unity element of the semiring in consideration). In the diagram, there are also arrows that end in a state, but do not start in one. These arrows are always labelled with an element of the semiring $(\mathbb{N}, +, \cdot, 0, 1)$. If q is a state that has an arrow "from nowhere" with label s entering it, this means that the initial weight of q is s. Similarly, if q is a state that has an arrow with label s leaving it "going nowhere", then s is the terminal weight of q. On the other hand, if there is no arrow "from nowhere" entering q, then the initial weight of q is 0 (the zero element of the semiring in consideration). The same holds for arrows "going nowhere" and terminal weights. In a similar manner, we shall draw diagrams of two-mode alternating weighted automata over an arbitrary commutative semiring.

There are two types of states in the automaton $\mathcal{A}_1$. The states with "+" label are called "sum states", while the states with "×" label are called "product states". For every state p of $\mathcal{A}_1$, we define a formal power series $|\mathcal{A}_1|_p$. If p is a sum state, then the definition of the power series $|\mathcal{A}_1|_p$ is the same as it was defined for (nonalternating) weighted automata; the coefficient of ε in $|\mathcal{A}_1|_p$ is equal to the terminal weight of p and for every c in Σ_1 and w in Σ_1^*, the coefficient of cw in $|\mathcal{A}_1|_p$ is $\sum_{t \in T(p,c)} weight(t)(|\mathcal{A}_1|_{ter(t)}, w)$, where $T(p, c)$ is the set of all transitions that start at p and are labelled with c.[1] If p is a product state, the power series $|\mathcal{A}_1|_p$ is defined as follows: the coefficient of ε in $|\mathcal{A}_1|_p$ is equal to the terminal weight of p and for each c in Σ_1 and w in Σ_1^*, the coefficient of cw in $|\mathcal{A}_1|_p$ is $\prod_{t \in T(p,c)} weight(t)(|\mathcal{A}_1|_{ter(t)}, w)$. In case $T(p, c)$ is empty, we define $(|\mathcal{A}_1|_p, w) = 0$. The reason why the states of a two-mode alternating weighted automaton are called sum states and product states is now evident: a sum state performs addition to calculate the coefficient of a word w in $|\mathcal{A}_1|$, while a product state performs multiplication to do this. The behaviour of the two-mode alternating weighted automaton $\mathcal{A}_1$ is defined in the same way as it was defined for (nonalternating) weighted automata, i.e., for every w in Σ_1^*, we have $(|\mathcal{A}_1|, w) = \sum_{p \in Q} initial\text{-}weight(p)(|\mathcal{A}_1|_p, w)$, where Q is the set of all states of $\mathcal{A}_1$.

Let us finally examine the behaviour of the automaton $\mathcal{A}_1$. Clearly, we have $(|\mathcal{A}_1|, \varepsilon) = 1$ and $(|\mathcal{A}_1|, b) = 0$. The reader can easily check that for every w in Σ_1^*, we have $(|\mathcal{A}_1|, aw) =$

[1] For simplicity, all the examples that we give in this section describe constructions of automata without ε-labelled transitions. This makes the definition of behaviour of two-mode alternating weighted automata that much simpler as well.

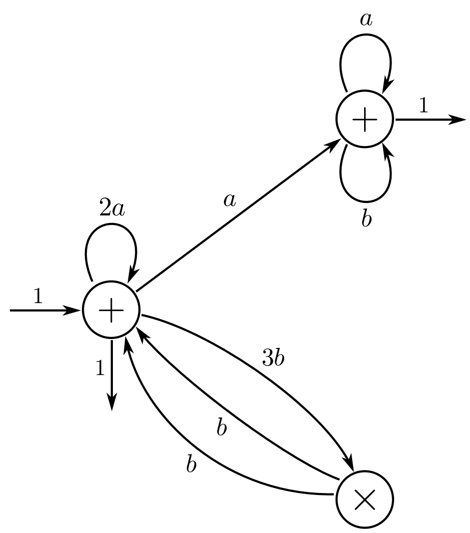

Figure 2.1: The automaton $\mathcal{A}_1$.

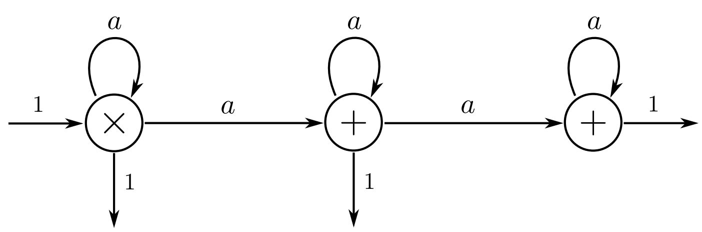

Figure 2.2: The automaton $\mathcal{A}_2$.

$2(|\mathcal{A}_1|, w) + 1$, $(|\mathcal{A}_1|, bbw) = 3(|\mathcal{A}_1|, w)^2$ and $(|\mathcal{A}_1|, baw) = 0$. These relations fully describe the formal power series $|\mathcal{A}_1|$.

Example 2.1.2. In Figure 2.2, we depict a two-mode alternating weighted automaton $\mathcal{A}_2$ over the alphabet $\Sigma_2 = \{a\}$ and over the semiring of natural numbers with standard operations of addition and multiplication. For every nonnegative integer n, the coefficient of a^n in $|\mathcal{A}_2|$ is $n!$.

Example 2.1.3. In Figure 2.3, we depict a two-mode alternating weighted automaton $\mathcal{A}_3$ over the alphabet $\Sigma_3 = \{\hat{1}\}$ and over a commutative semiring containing an element s. For every nonnegative integer n, we have $(|\mathcal{A}_3|, (\hat{1})^n) = s^{2^n}$.

Example 2.1.4. We can make the previous example more general. In Figure 2.4, we depict a two-mode alternating weighted automaton $\mathcal{A}_4$ over an alphabet $\Sigma_4 = \{\hat{0}, \hat{1}\}$ and over a commutative semiring containing an element s. From now on, we shall often label some arrows in diagrams

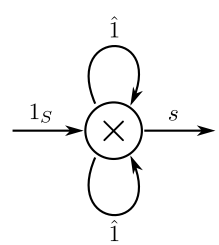

Figure 2.3: The automaton $\mathcal{A}_3$.

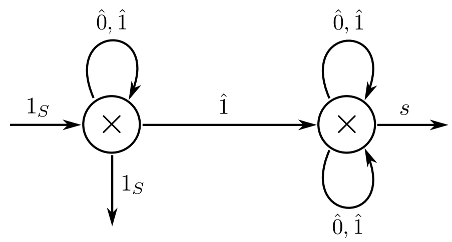

Figure 2.4: The automaton $\mathcal{A}_4$.

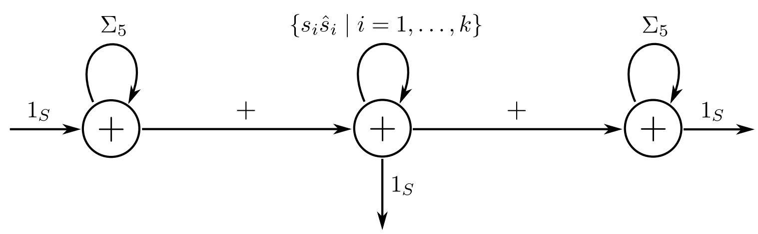

Figure 2.5: The automaton $\mathcal{A}_5$.

of two-mode alternating weighted automata with a set of symbols or with a set of symbols with coefficients. If an arrow leading from state p to state q is labelled with a set of symbols, then there is a transition from p into q with label a and weight 1_S for every symbol a in this set. If this arrow is labelled with a set of symbols with coefficients, then there is a transition from p into q with label a and weight s for every symbol a with coefficient s in this set. If we want to label an arrow with a set $\{d_1, d_2, \ldots, d_m\}$, we often simply label it with elements $d_1, d_2, \ldots, d_m$ separated by commas, omitting the braces. We can see this practice being used also in Figure 2.4.

Every word w over $\Sigma_4 = \{\hat{0}, \hat{1}\}$ can be viewed as a binary representation of some nonnegative integer n; we shall write $int(w)$ to denote this nonnegative integer n. The reader can check that for every w in Σ_4^*, the coefficient of w in $|\mathcal{A}_4|$ is $s^{int(w)}$.

Example 2.1.5. We shall now give an example of a (nonalternating) weighted automaton, which we shall later modify into a two-mode alternating weighted automaton. Figure 2.5 depicts a weighted automaton $\mathcal{A}_5$ over a commutative semiring S and over an alphabet Σ_5; for some particular elements $s_1, s_2, \ldots, s_k$ of S, the alphabet Σ_5 consists of the symbol $+$ and symbols $\hat{s}_i$ for $i = 1, \ldots, k$.

Let us examine the behaviour of the automaton $\mathcal{A}_5$. Let L' consist of all words w over Σ_5 such that $w = \hat{s}_{i_1}\hat{s}_{i_2}\ldots\hat{s}_{i_m}$, where m is a nonnegative integer and $i_1, \ldots, i_m$ are positive integers such that $i_j \leq k$ for $j = 1, \ldots, m$. For each such w, we define $elem(w)$ to be the semiring element $\prod_{j=1}^{m} s_{i_j}$. Let L_{elem} consist of all words w over Σ_5 such that $w = +u_1 + u_2 \ldots + u_m$, where m is a nonnegative integer and u_i is in L' for $i = 1, \ldots, m$. For each such w, we define $elem(w) = \sum_{i=1}^{m} elem(u_i)$. It is easy to see that $(|\mathcal{A}_5|, w) = elem(w)$ for every w in L_{elem}. The reader might want to try to determine the coefficient of w in $|\mathcal{A}_5|$ if w is in $\Sigma_5^* - L_{elem}$.

Example 2.1.6. We shall now modify the (nonalternating) weighted automaton $\mathcal{A}_5$ into a two-mode alternating weighted automaton $\mathcal{A}_6$ over S and over the alphabet $\Sigma_6 = \Sigma_5 \cup \{x\}$. This new automaton is depicted in Figure 2.6. We can see that $(|\mathcal{A}_6|, x^m u) = (elem(u))^m$ for each nonnegative integer m and each word u in L_{elem}. The automaton $\mathcal{A}_6$ has the ability to calculate exponents of semiring elements.

Example 2.1.7. We can go even further with the previous example. Let $\Sigma_7 = \Sigma_6 \cup \{\oplus, \#\}$. Let L'' consist of all words w over Σ_7 such that $w = \oplus x^{m_1} \oplus x^{m_2} \ldots \oplus x^{m_l}$ for some nonnegative integers l and $m_1, \ldots, m_l$. For each such w, let the polynomial $\sum_{i=1}^{l} x^{m_i}$ in $S[x]$ be denoted by

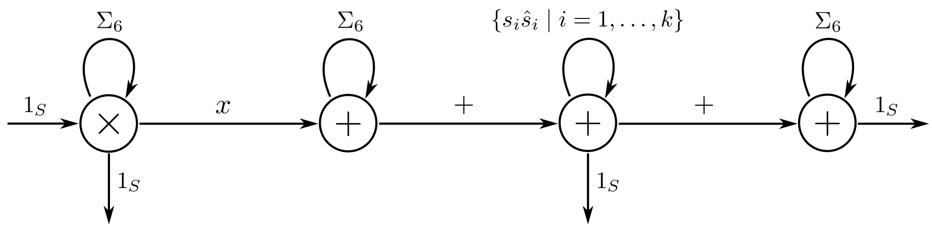

Figure 2.6: The automaton $\mathcal{A}_6$.

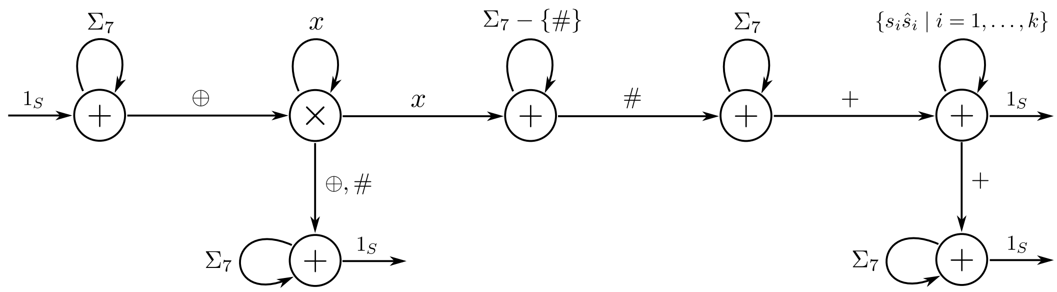

Figure 2.7: The automaton $\mathcal{A}_7$.

$poly[w]$. In Figure 2.7, we depict a two-mode alternating weighted automaton $\mathcal{A}_7$ over S and Σ_7 such that $(|\mathcal{A}_7|, u\#v) = poly[u](elem(v))$ for each u in L'' and v in L_{elem}.

Example 2.1.8. To make the previous example even more general, let $\Sigma_8 = \Sigma_7 \cup \{\langle, \rangle\}$ (i.e., Σ_8 consists of symbols from Σ_7 and symbols for angle brackets). Let L_{poly} consist of all words w over Σ_8 such that $w = \oplus\langle u_1\rangle x^{m_1} \oplus \langle u_2\rangle x^{m_2} \ldots \oplus \langle u_l\rangle x^{m_l}$ for some nonnegative integers l and $m_1, \ldots, m_l$ and words $u_1, \ldots, u_l$ in L_{elem}. For each such w, let the polynomial $\sum_{i=1}^{l} elem(u_i)x^{m_i}$ in $S[x]$ be denoted by $poly[w]$. In Figure 2.8, we depict a two-mode alternating weighted automaton $\mathcal{A}_8$ over S and Σ_8 such that $(|\mathcal{A}_8|, u\#v) = poly[u](elem(v))$ for each u in L_{poly} and v in L_{elem}. The automaton $\mathcal{A}_8$ has the ability to substitute into polynomials.

2.2 Definition of Alternating Weighted Automata

We shall now state the key definition of this thesis – the definition of alternating weighted automata. We have already anticipated that we shall give two alternative definitions of alternating weighted automata. In one of these definitions, every state is either a "sum" state or a "product" state, where a sum state can perform addition only and a product state can perform multiplication only. We shall first give the more general definition, in which every state can perform both addition and multiplication.

Definition 2.2.1. Let S be a commutative semiring. An *alternating weighted automaton* over S is a tuple $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$, where Q is a nonempty finite set of states with $n := |Q|$; Σ is an alphabet; $\psi : (Q \times \Sigma) \to S[x_1, \ldots, x_n]_{\text{const}=0}$ is a polynomial assigning function; P_0 in $S[x_1, \ldots, x_n]_{\text{const}=0}$ is an initial polynomial; $\tau : Q \to S$ is a terminal weighting function.

We shall always assume that the states of each alternating weighted automaton are somehow linearly ordered. Although this linear ordering is not part of the definition above, it is nevertheless important, as we shall see in the following definition. Moreover, whenever we refer to "the i-th state", where i is a positive integer, we mean the i-th state with respect to the linear ordering of the states of the automaton in consideration.

For every alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$, every p in Q, and a in Σ, the polynomial $\psi[p, a]$ can be denoted also by $\psi[i, a]$ if p is the i-th state of $\mathcal{A}$. In some cases, we

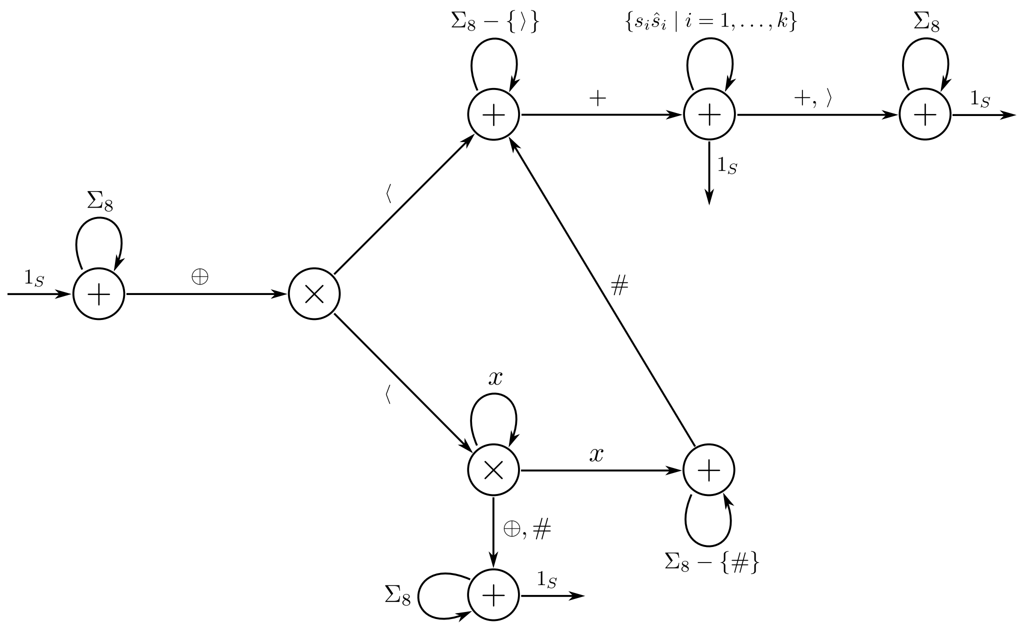

Figure 2.8: The automaton $\mathcal{A}_8$.

shall also write $\tau(i)$ instead of $\tau(p)$. We shall shortly introduce also some other notation, in which states are interchangeable with their numerical order.

Definition 2.2.2. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over a commutative semiring S, let $n = |Q|$. For every p in Q, we define a formal power series $|\mathcal{A}|_p$ (also denoted by $|\mathcal{A}|_i$ if p is the i-th state) in $S\langle\langle\Sigma^*\rangle\rangle$ as follows:

1. The coefficient of ε in $|\mathcal{A}|_p$ is defined by
$$(|\mathcal{A}|_p, \varepsilon) = \tau(p).$$
2. For each a in Σ and w in Σ^*, the coefficient of aw in $|\mathcal{A}|_p$ is defined by
$$(|\mathcal{A}|_p, aw) = \psi[p, a]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle\Sigma^*\rangle\rangle$ defined for all w in Σ^* by
$$(|\mathcal{A}|, w) = P_0\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

Let us now discuss the relationship between alternating weighted automata on the one hand, and Boolean alternating automata and (nonalternating) weighted automata on the other hand. First, a Boolean alternating automaton can be viewed as an alternating weighted automaton over the Boolean semiring and vice versa. The only difference between the definitions of these two models is that one is defined with positive Boolean formulae and the other is defined with polynomials over the Boolean semiring. However, this difference is trivial due to the obvious correspondence between positive Boolean formulae and polynomials over the Boolean semiring.

We shall now look at the relationship between alternating weighted automata and (nonalternating) weighted automata. Let S be a commutative semiring and $\mathcal{L}$ be the set of all polynomials of the form $c_1x_1 + c_2x_2 + \ldots + c_nx_n$, where n is a nonnegative integer, c_i is in S, and x_i is an indeterminate for $i = 1, \ldots n$. An alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ over S shall be called "sum-only" if the polynomial P_0 is in $\mathcal{L}$ and so are the polynomials $\psi[p, a]$ for every p in Q and a in Σ. One can easily see that a (nonalternating) weighted automaton without ε-labelled transitions can be viewed as a sum-only alternating weighted automaton and vice versa.

2.3 Evaluation Polynomials

Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton, let $n := |Q|$. Assume that a word $w = a_1a_2 \dots a_m$ is given, where $a_1, \dots, a_m$ are in Σ, and that our task is to calculate the coefficient of w in $|\mathcal{A}|$. One obvious approach is the "bottom-up evaluation". We start with the values $\tau(p)$ for each p in Q. Substituting these values into the polynomial $\psi[p, a_m]$, we can evaluate $(|\mathcal{A}|_p, a_m)$ for each p in Q. If we substitute these values into the polynomial $\psi[p, a_{m-1}]$, we can evaluate $(|\mathcal{A}|_p, a_{m-1}a_m)$ for each p in Q. By repeating this process, we are eventually able to evaluate $(|\mathcal{A}|_p, a_1a_2 \dots a_m)$. If we now substitute these values into the polynomial P_0, we obtain the weight of the word w.

Another approach to calculate the weight of the word w is the "top-down" method. We start with the polynomial P_0. Substituting $x_i = \psi[i, a_1]$ for $i = 1, \dots, n$ into the polynomial P_0, we obtain a polynomial P_1. In the next step, we substitute $x_i = \psi[i, a_2]$ for $i = 1, \dots, n$ into the polynomial P_1 and obtain a polynomial P_2. This process is repeated, successively constructing polynomials $P_3, P_4, \dots$, until the polynomial P_m is constructed. If we now substitute $x_i = \tau(i)$ for $i = 1, \dots, n$ into the polynomial P_m, we obtain the weight of the word w. The same approach can be used if one wishes to calculate the coefficient of the word w in $|\mathcal{A}|_i$ for some i in $\{1, \dots, n\}$. The only difference is that in this case, we start with the polynomial $R_0 = x_i$ instead of the polynomial P_0, constructing polynomials $R_1, R_2, \dots, R_m$ in the following steps. The polynomials $P_0, P_1, \dots, P_m$ and $R_0, R_1, \dots, R_m$ shall be called *evaluation polynomials*.

Definition 2.3.1. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over a commutative semiring S, let $n = |Q|$. For every p in Q and w in Σ^*, we define the *evaluation polynomial* $P_\mathcal{A}[p, w]$ (also denoted by $P_\mathcal{A}[i, w]$ if p is the i-th state in $\mathcal{A}$) in $S[x_1, \dots, x_n]$ as follows:

1. If p is the i-th state of $\mathcal{A}$, then
$$P_\mathcal{A}[p, \varepsilon] = x_i.$$
2. If a is in Σ and w is in Σ^*, then
$$P_\mathcal{A}[p, wa] = P_\mathcal{A}[p, w]\big(\psi[1, a], \psi[2, a], \dots \psi[n, a]\big).$$

For every w in Σ^*, we define the *evaluation polynomial* $P_\mathcal{A}[w]$ in $S[x_1, \dots, x_n]$ by

$$P_\mathcal{A}[w] = P_0\big(P_\mathcal{A}[1, w], P_\mathcal{A}[2, w], \dots, P_\mathcal{A}[n, w]\big).$$

The following lemma shows the usefulness of evaluation polynomials. The proof of this claim is left to the reader.

Lemma 2.3.2. *Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton, let $n = |Q|$. If v, w are in Σ^* and p is in Q, then*

$$(|\mathcal{A}|_p, vw) = P_\mathcal{A}[p, v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \dots, (|\mathcal{A}|_n, w)\big),$$
$$(|\mathcal{A}|, vw) = P_\mathcal{A}[v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \dots, (|\mathcal{A}|_n, w)\big).$$

2.4 Two-Mode Alternating Weighted Automata

In this section, we shall give a definition of *two-mode alternating weighted automata*. This definition can also be viewed as an alternative definition of alternating weighted automata; although these two models are not formally identical, they are the same in their essence. The main difference between alternating weighted automata and two-mode alternating weighted automata is that the latter has two types of states: "sum" states, which can only perform addition, and "product" states, which can only perform multiplication. Another feature that sets two-mode alternating weighted automata apart from alternating weighted automata is the presence of ε-labelled transitions. The definition of two-mode alternating weighted automata is actually very similar to the definition of (nonalternating) weighted automata given in the previous chapter.

Definition 2.4.1. A *two-mode alternating weighted automaton* over a commutative semiring S is a tuple $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$, where $Q^{\oplus}, Q^{\otimes}$ are finite sets of states with $Q^{\oplus} \cup Q^{\otimes} \neq \emptyset$ and $Q^{\oplus} \cap Q^{\otimes} = \emptyset$; Σ is an alphabet; T is a finite set of transitions, with which we associate maps $init, ter : T \to (Q^{\oplus} \cup Q^{\otimes})$ and $\sigma : T \to \Sigma \cup \{\varepsilon\}$; $\nu : T \to S$ is a transition weighting function; $\iota : (Q^{\oplus} \cup Q^{\otimes}) \to S$ is an initial weighting function; $\tau : (Q^{\oplus} \cup Q^{\otimes}) \to S$ is a terminal weighting function.

A two-mode alternating weighted automaton can be viewed as a directed multigraph with labelled edges, where $Q^{\oplus} \cup Q^{\otimes}$ is the set of vertices and T is the set of edges, while $init, ter$, and σ are the maps that assign initial vertex, terminal vertex and label to each edge, respectively. The elements of $Q^{\oplus}$ and $Q^{\otimes}$ shall be called *sum states* and *product states*, respectively.

Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton, let p be in $Q^{\oplus} \cup Q^{\otimes}$ and a be in $\Sigma \cup \{\varepsilon\}$. We shall write $T_{\mathcal{A}}(p)$ to denote the set of all transitions t in T such that $init(t) = p$, while $T_{\mathcal{A}}(p, a)$ shall be used to denote the set of all transitions t in T satisfying $init(t) = p$ and $\sigma(t) = a$. We shall write $T^{\varepsilon}_{\mathcal{A}}$ to denote the set of all ε-labelled transitions of $\mathcal{A}$. If clear from the context, we shall often omit the subscript denoting the automaton in consideration.

The behaviour of a two-mode alternating weighted automaton will be defined only if the automaton contains no cycles of ε-labelled transitions.

Definition 2.4.2. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton. We say that $\mathcal{A}$ is *without ε-cycles* if the directed graph with vertex set $Q^{\oplus} \cup Q^{\otimes}$ and edge set T^{ε} contains no cycle.

From now on, whenever we refer to a two-mode alternating weighted automaton, we shall always mean a two-mode alternating weighted automaton without ε-cycles. We are now prepared to define the behaviour of a two-mode alternating weighted automaton.

Definition 2.4.3. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton without ε-cycles over a commutative semiring S. For each p in $Q^{\oplus} \cup Q^{\otimes}$, we define a formal power series $|\mathcal{A}|_p$ in $S\langle\langle \Sigma^* \rangle\rangle$ as follows:

1. If p is in $Q^{\oplus}$, then
   (a) the coefficient of ε in $|\mathcal{A}|_p$ is defined by
   $$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, \varepsilon);$$
   (b) for each a in Σ and w in Σ^*,
   $$(|\mathcal{A}|_p, aw) = \sum_{t \in T(p,a)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, w) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, aw).$$
2. If p is in $Q^{\otimes}$, then
   (a) if $T(p, \varepsilon) \neq \emptyset$, then
   $$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \prod_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, \varepsilon),$$
   otherwise
   $$(|\mathcal{A}|_p, \varepsilon) = \tau(p);$$
   (b) for each a in Σ and w in Σ^*, if $T(p, a) \cup T(p, \varepsilon) \neq \emptyset$, then
   $$(|\mathcal{A}|_p, aw) = \prod_{t \in T(p,a)} \Big(\nu(t) \cdot (|\mathcal{A}|_{ter(t)}, w)\Big) \cdot \prod_{t \in T(p,\varepsilon)} \Big(\nu(t) \cdot (|\mathcal{A}|_{ter(t)}, aw)\Big),$$
   otherwise
   $$(|\mathcal{A}|_p, aw) = 0.$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle\Sigma^*\rangle\rangle$ defined for all $w \in \Sigma^*$ by

$$(|\mathcal{A}|, w) := \sum_{p \in Q^{\oplus} \cup Q^{\otimes}} \iota(p) \cdot (|\mathcal{A}|_p, w).$$

Another way to describe the behaviour of a two-mode alternating weighted automaton is through the notion of a *run tree*. If a run in a two-mode alternating weighted automaton is in some state q, there might be multiple transitions, which the run might follow. If q is a sum state, only one of these transitions is followed, but if q is a product state, all of them are followed. So the flow of a run in a two-mode alternating weighted automaton can be viewed as a rooted tree, which branches whenever it passes a product state. A run tree is supposed to represent the flow of a run, capturing all states and transitions it passes.

We shall now give a formal definition. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over a semiring S, let w be a word in Σ^*. We shall construct a run tree R on the word w as a directed rooted tree with node set V and edge set E, node labelling $state : V \to Q^{\oplus} \cup Q^{\otimes}$ and edge labelling $transition : E \to T \cup \{start, finish, terminate\}$. The *state* and *transition* labels represent the current state and transition, respectively. If x and y are two nodes in R that are connected by an edge from x to y, we shall write xy to denote this edge. We also want to capture the start and the ends of a run in its run tree; we shall use directed half-edges for this purpose (a half-edge is an edge with only one end). The start of a run will be represented by a half-edge entering the root of R. To represent the ends of a run, every leaf of R will have an half-edge leaving it. These half-edges shall be called a start-edge and finish-edges, respectively and we shall include them in the edge set E as well. We require that each node has at most one start-edge and at most one finish-edge. Moreover, a node x has a start-edge iff x is a root and a finish-edge iff x is a leaf. The *transition* label of a start-edge is always $start$, while a finish-edge is always labelled either with $finish$ or with *terminate*. The $finish$ label symbolizes the fact that a particular branch in the run tree has successfully ended. On the other hand, the *terminate* label symbolizes the fact that a particular branch in the run tree has ended prematurely, because the automaton got "stuck".

We shall define run trees through the notion of "partial" run trees. A partial run tree R is a directed rooted tree with node labelling *state*, edge labelling *transition*, and finish-edges leaving the leaves of R. It has the same structure as an "ordinary" run tree as described in the previous paragraph with the single difference that its root has no start-edge. For each p in $Q^{\oplus} \cup Q^{\otimes}$ and w in Σ^*, we define a class $\mathscr{R}(p, w)$ of partial run trees. A tree R is in $\mathscr{R}(p, \varepsilon)$ iff the *state* label of its root x is p and one of the following conditions is satisfied:

1. The root x of R has no children and the *transition* label of its finish-edge is $finish$.
2. The state p is in $Q^{\oplus}$, x has exactly one child y and there exists t in $T(p, \varepsilon)$ such that $transition(xy) = t$ and the subtree of R with root in y is in $\mathscr{R}(ter(t), \varepsilon)$.
3. The state p is in $Q^{\otimes}$ and there exists a 1-1 correspondence between transitions in nonempty set $T(p, \varepsilon)$ and the children of x such that the following is satisfied for each t in $T(p, \varepsilon)$: if y is the child of x that corresponds to t, then $transition(xy) = t$ and the subtree of R with root in y is in $\mathscr{R}(ter(t), \varepsilon)$.

Let $w = av$, where a is in Σ and v is in Σ^*. A tree R is in $\mathscr{R}(p, w)$ iff the *state* label of its root x is p and one of the following conditions is satisfied:

1. The set $T(p, a) \cup T(p, \varepsilon)$ is empty, the root x has no children, and the *transition* label of the finish-edge of x is *terminate*.
2. The state p is in $Q^{\oplus}$, x has exactly one child y, and there exists t in $T(p, \varepsilon) \cup T(p, a)$ such that $transition(xy) = t$ and the subtree of R with root in y is in $\mathscr{R}(ter(t), u)$, where u is a word in Σ^* such that $\sigma(t)u = w$.

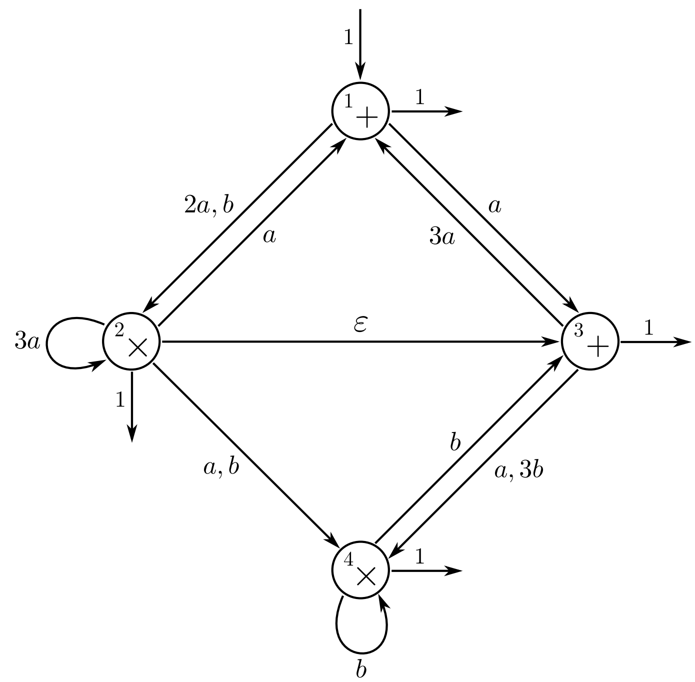

Figure 2.9: The two-mode alternating weighted automaton $\mathcal{A}$ from Example 2.4.1.

3. The state p is in $Q^{\otimes}$ and there exists a 1-1 correspondence between transitions in nonempty set $T(p,\varepsilon) \cup T(p,a)$ and the children of x such that the following is satisfied for each t in $T(p,\varepsilon) \cup T(p,a)$: if y is the child of x that corresponds to t, then $\mathit{transition}(xy) = t$ and the subtree of R with root in y is in $\mathscr{R}(\mathit{ter}(t), u)$, where u is a word in Σ^* such that $\sigma(t)u = w$.

A tree R is a *run tree* on a word w in Σ^* iff there exists a state p in $Q^{\oplus} \cup Q^{\otimes}$ such that the following two conditions are satisfied:

1. The start edge of the root of R is labelled with start.

2. The partial run tree, which is obtained from R if we remove the start-edge entering its root, is in $\mathscr{R}(p, w)$.

The set of all run trees on w shall be denoted by $\mathscr{R}(w)$.

Example 2.4.1. In Figure 2.9, we give an example of a two-mode alternating weighted automaton $\mathcal{A}$ over the alphabet $\Sigma = \{a, b\}$ and over the semiring $\mathbb{R}$ of real numbers with standard operations of addition and multiplication. The set of states of $\mathcal{A}$ is $\{1, 2, 3, 4\}$.

In Figure 2.10, we depict all run trees on aab in the automaton $\mathcal{A}$. In each of the depicted run trees, each node is labelled with a number representing the *state* label of the node. Each node is also labelled either with "+" or with "×" – this label indicates, whether the state corresponding to the given node is a sum state or a product state. Each edge in the depicted run trees is labelled with a symbol from Σ, which might be preceded by a coefficient from $\mathbb{R}$ – the symbol and the coefficient represent the label and the weight of the transition of $\mathcal{A}$ corresponding to this edge. If the symbol is not preceded by a coefficient, then the weight of the corresponding transition is 1.[2]

Every run in a two-mode alternating weighted automaton $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ has a weight that depends on the transitions it passes. For each edge e in a run tree R, we define

[2]By the definition above, each edge in a run tree is labelled with a transition (or with start, finish, or $\mathit{terminate}$). In this example, we have nevertheless decided to label the edges with symbols and weights of corresponding transitions instead, because we believe that this makes the example easier to understand.

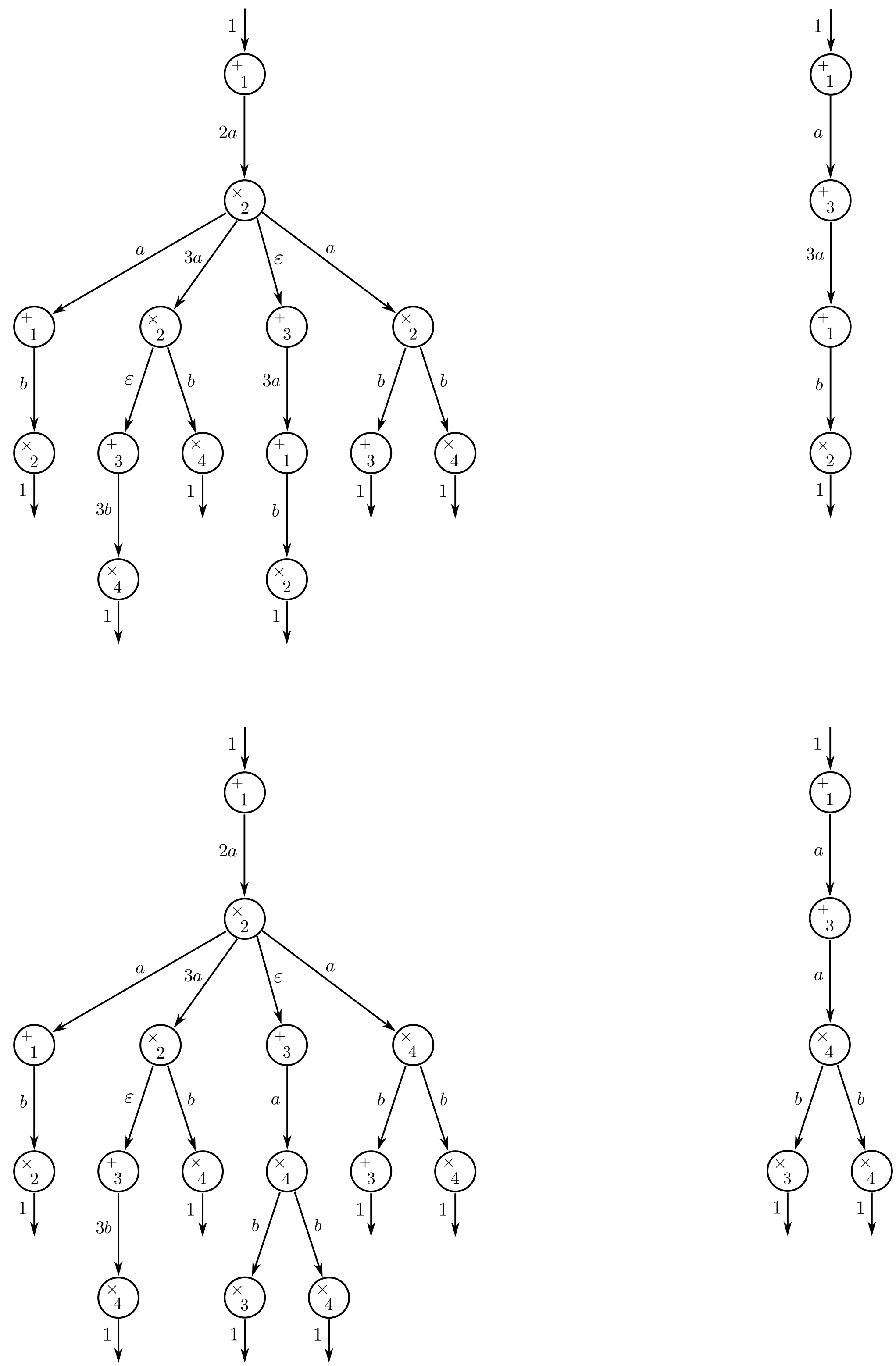

Figure 2.10: All run trees on aab in the two-mode alternating weighted automaton $\mathcal{A}$ from Example 2.4.1.

its weight *weight*(e) as follows. If e is an "ordinary" edge (i.e. an edge with two ends) and *transition*$(e) = t$, where t is in T, then *weight*$(e) = \nu(t)$. If e is a start-edge entering the root x of R and *state*$(x) = p$, then *weight*$(e) = \iota(p)$. Similarly, if e is a finish-edge leaving a leaf y of R, *transition*$(e) =$ *finish*, and *state*$(y) = p$, then *weight*$(e) = \tau(p)$. Finally, if e is a finish-edge and its *transition* label is *terminate*, then *weight*$(e) = 0$. We now define the weight *weight*(R) of a run tree R with edge set E to be the product of weights of all edges in E, i.e., $weight(R) = \prod_{e\in E} weight(e)$. Similarly, if R is a partial run tree with edge set E, we define $weight(R) = \prod_{e\in E} weight(e)$. The following theorem characterizes the behaviour of an alternating weighted automaton through the notion of a run tree.

Theorem 2.4.4. *Let $\mathcal{A}$ be a two-mode alternating weighted automaton over a commutative semiring S and over an alphabet Σ. Then for each w in Σ^*,*

$$(|\mathcal{A}|, w) = \sum_{R\in\mathscr{R}(w)} weight(R).$$

Proof sketch. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over a commutative semiring S. We shall prove that

$$(|\mathcal{A}|_p, w) = \sum_{R\in\mathscr{R}(p,w)} weight(R) \tag{2.1}$$

for every p in $Q^{\oplus} \cup Q^{\otimes}$ and w in Σ^*. The proof shall be done by mathematical induction on *depth*(p, w), which we define to be the maximal depth of all run trees in $\mathscr{R}(p, w)$.

Let $depth(p, w) = 0$ and $w = \varepsilon$. The set $T(p, \varepsilon)$ is empty (for otherwise $depth(p, \varepsilon)$ would be greater than zero). Therefore, $\mathscr{R}(p, \varepsilon) = \{R_0\}$, where R_0 is a partial run tree that consists of a single node x and the weight of the finish-edge of x is $\tau(p)$. We have $\sum_{R\in\mathscr{R}(p,\varepsilon)} weight(R) = \tau(p) = (|\mathcal{A}|_p, \varepsilon)$.

Let $depth(p, w) = 0$ and $w = av$, where a is in Σ and v is in Σ^*. The set $T(p, \varepsilon) \cup T(p, a)$ is empty (for otherwise $depth(p, w)$ would be greater than zero). Therefore, $\mathscr{R}(p, w) = \{R_0\}$, where R_0 is a partial run tree that consists of a single node x and the weight of the finish-edge of x is 0. We have $\sum_{R\in\mathscr{R}(p,w)} weight(R) = 0 = (|\mathcal{A}|_p, w)$.

Let $depth(p, w) > 0$, $w = \varepsilon$ and p be in $Q^{\oplus}$. It is quite easy to see that

$$\sum_{R\in\mathscr{R}(p,\varepsilon)} weight(R) = weight(R_0) + \sum_{t\in T(p,\varepsilon)} \Big(\nu(t) \cdot \sum_{R\in\mathscr{R}(ter(t),\varepsilon)} weight(R)\Big),$$

where R_0 is a partial run tree that consists of a single node x and the weight of the finish-edge of x is $\tau(p)$. By the induction hypothesis, we thus have

$$\sum_{R\in\mathscr{R}(p,\varepsilon)} weight(R) = \tau(p) + \sum_{t\in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, \varepsilon) = (|\mathcal{A}|_p, \varepsilon).$$

Let $depth(p, w) > 0$, $w = \varepsilon$ and p be in $Q^{\otimes}$. It is quite easy to see that

$$\sum_{R\in\mathscr{R}(p,\varepsilon)} weight(R) = weight(R_0) + \prod_{t\in T(p,\varepsilon)} \Big(\nu(t) \cdot \sum_{R\in\mathscr{R}(ter(t),\varepsilon)} weight(R)\Big),$$

where R_0 is a partial run tree that consists of a single node x and the weight of the finish-edge of x is $\tau(p)$. By the induction hypothesis, we have

$$\sum_{R\in\mathscr{R}(p,w)} weight(R) = \tau(p) + \prod_{t\in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, \varepsilon) = (|\mathcal{A}|_p, \varepsilon).$$

We are left with two more cases. The first case is when $depth(p, w) > 0$, w is nonempty, and p is in $Q^{\oplus}$ and the second case is when $depth(p, w) > 0$, w is nonempty, and p is in $Q^{\otimes}$. One can deal with these cases in pretty much the same way as with the previous two cases. We leave this for the reader.

We have thus proved that (2.1) holds for every p in $Q^\oplus \cup Q^\otimes$ and w in Σ^*. For every w in Σ^*, we have

$$\begin{aligned}(|\mathcal{A}|, w) &= \sum_{p \in Q^\oplus \cup Q^\otimes} \iota(p) \cdot (|\mathcal{A}|_p, w) = \\ &= \sum_{p \in Q^\oplus \cup Q^\otimes} \Big(\iota(p) \cdot \sum_{R \in \mathscr{R}(p,w)} weight(R)\Big) = \\ &= \sum_{R \in \mathscr{R}(w)} weight(R).\end{aligned}$$

The theorem is proved. □

Evaluation polynomials, which we have defined for alternating weighted automata, can be defined for two-mode alternating weighted automata as well. The definition of evaluation polynomials is valid only in connection with some linear ordering of the states of the automaton in consideration. For this reason, we shall always assume that the set of states of each two-mode alternating weighted automaton is already somehow linearly ordered. Whenever we refer to "the i-th state", we mean the i-th state with respect to this linear ordering. It is also sometimes convenient to replace a state with its numerical order in our notation. For example, if $\mathcal{A} = (Q^\oplus, Q^\otimes, \Sigma, T, \nu, \iota, \tau)$ is a two-mode alternating weighted automaton and p is its i-th state, then we can write $|\mathcal{A}|_i$, $\iota(i)$, and $\tau(i)$ instead of $|\mathcal{A}|_p$, $\iota(p)$, and $\tau(p)$, respectively.

Definition 2.4.5. Let $\mathcal{A} = (Q^\oplus, Q^\otimes, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over a commutative semiring S, let $n = |Q^\oplus \cup Q^\otimes|$. For each w in Σ^* and p in $Q^\oplus \cup Q^\otimes$, we define the *evaluation polynomial* $P_\mathcal{A}[p, w]$ (also denoted by $P_\mathcal{A}[i, w]$ if p is the i-th state of $\mathcal{A}$) in $S[x_1, \ldots, x_n]$ as follows:

1. If p is the i-th state in $Q^\oplus \cup Q^\otimes$, then
$$P_\mathcal{A}[p, \varepsilon] = x_i.$$

2. If p is in $Q^\oplus$, a is in Σ, and w is in Σ^*, then
$$P_\mathcal{A}[p, aw] = \sum_{t \in T(p,a)} \nu(t) \cdot P_\mathcal{A}[ter(t), w] + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot P_\mathcal{A}[ter(t), aw].$$

3. If p is in $Q^\otimes$, a is in Σ, and w is in Σ^*, then
$$P_\mathcal{A}[p, aw] = \prod_{t \in T(p,a)} \Big(\nu(t) \cdot P_\mathcal{A}[ter(t), w]\Big) \cdot \prod_{t \in T(p,\varepsilon)} \Big(\nu(t) \cdot P_\mathcal{A}[ter(t), aw]\Big),$$
unless $T(p, a) \cup T(p, \varepsilon) = \emptyset$, in which case we define
$$P_\mathcal{A}[p, aw] = 0.$$

For each w in Σ^*, we define the *evaluation polynomial* $P_\mathcal{A}[w]$ in $S[x_1, \ldots, x_n]$ by

$$P_\mathcal{A}[w] = \sum_{p \in Q^\oplus \cup Q^\otimes} \iota(p) \cdot P_\mathcal{A}[p, w].$$

The following lemma is an analogy to Lemma 2.3.2. We leave the proof of this fact for the reader.

Lemma 2.4.6. *Let $\mathcal{A} = (Q^\oplus, Q^\otimes, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton, let $n = |Q^\oplus \cup Q^\otimes|$. If v, w are in Σ^* and p is in $Q^\oplus \cup Q^\otimes$, then*

$$(|\mathcal{A}|_p, vw) = P_\mathcal{A}[p, v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big),$$
$$(|\mathcal{A}|, vw) = P_\mathcal{A}[v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

We have already mentioned at the beginning of this section that the definition of two-mode alternating weighted automata can be viewed as just an alternative definition of alternating weighted automata. The following theorem justifies this claim.

Theorem 2.4.7. *A formal power series r over a commutative semiring S and over an alphabet Σ is realized by an alternating weighted automaton over S iff it is realized by a two-mode alternating weighted automaton over S.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over S. We shall construct a two-mode alternating weighted automaton $\mathcal{A}' = (Q^\oplus, Q^\otimes, \Sigma, T, \nu, \iota, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$.

For each p in Q and a in Σ, the polynomial $\psi[p, a]$ can be written as a sum of distinct nonzero monomials $m_1, m_2, \ldots, m_k$. Let $M(p, a) = \{m_i \mid i = 1, \ldots, k\}$ and $M = \bigcup_{p \in Q, a \in \Sigma} M(p, a)$. Similarly, the polynomial P_0 can be written as a sum of distinct nonzero monomials $m'_1, m'_2, \ldots, m'_l$. Let $M_0 = \{m'_i \mid i = 1, \ldots, l\}$.

First, let us construct the sets of states $Q^\oplus$ and $Q^\otimes$ by taking $Q^\oplus = Q$ and $Q^\otimes = M \cup M_0$. The set of transitions T shall be constructed with the following modus operandi (it will be followed also in some later constructions). The set T will consist of tuples, where the first entry of each tuple is the initial state, the second entry is the label, the third entry is the weight, and the fourth entry is the terminal state of the transition. The fifth entry, if present, is an index. This entry is present in the tuple representing a transition t only in case the constructed automaton contains transitions that are parallel with t and have the same label as t. Now that we have clarified this, we can return to the construction of T. This set will be constructed from the following two components:

- The set T_α that consists of all tuples (p, a, s, m), where p is in Q, a is in Σ, m is a monomial in $M(p, a)$, and $s = \mathit{coef}(m)$.
- The set T_β that consists of all tuples $(m, \varepsilon, 1_S, p, i)$, where m is in $M \cup M_0$, p is the j-th state in Q (with respect to the ordering of states of $\mathcal{A}$) and i is a positive integer such that $i \leq \mathit{exp}(m, j)$.

We now define $T = T_\alpha \cup T_\beta$. The maps $\mathit{init}, \mathit{ter}, \sigma$ and ν are defined by the corresponding entries of the transitions.

Let us now define the initial weighting function ι. If m is a monomial in M_0, we define $\iota(m) = \mathit{coef}(m)$. For every other state p in $Q^\oplus \cup Q^\otimes$, we define $\iota(p) = 0$. Finally, let us define the terminal weighting function τ'. If p is in Q, we define $\tau'(p) = \tau(p)$. For every other state p in $Q^\oplus \cup Q^\otimes$, we define $\tau'(p) = 0$.

We have now fully defined the two-mode alternating weighted automaton $\mathcal{A}'$. It is not too hard to show that $|\mathcal{A}'| = |\mathcal{A}|$. We shall omit the proof.

To prove the converse implication, let $\mathcal{A} = (Q^\oplus, Q^\otimes, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over S, let $n = |Q^\oplus \cup Q^\otimes|$. We shall construct an alternating weighted automaton $\mathcal{A}' = (Q, \Sigma, \psi, P_0, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$.

Let $Q = Q^\oplus \cup Q^\otimes$ and let Q keep the linear ordering of $Q^\oplus \cup Q^\otimes$. The initial polynomial P_0 is defined by $P_0 = \sum_{i=1}^{n} \iota(i) x_i$. For each p in Q and a in Σ, we define $\psi[p, a] = P_\mathcal{A}[p, a]$. The terminal weighting function τ' is defined by $\tau'(p) = (|\mathcal{A}|_p, \varepsilon)$ for each p in Q.

The reader can easily show that $|\mathcal{A}'| = |\mathcal{A}|$. We shall omit the proof of this fact as well. □

2.5 Elimination of ε-Labelled Transitions

Unlike in the case of alternating weighted automata, the definition of two-mode alternating weighted automata allows ε-labelled transitions. It would have been possible to allow ε-labelled transitions also in the definition of the former, but we believe that they would not be particularly useful. On the other hand, ε-labelled transitions have already proved to be handy when constructing two-mode alternating weighted automata. We shall now show that they are nevertheless not strictly necessary in the definition of two-mode alternating weighted automata.

Definition 2.5.1. A two-mode alternating weighted automaton $\mathcal{A}$ is *ε-free* if it contains no ε-labelled transitions.

Theorem 2.5.2. *For every two-mode alternating weighted automaton $\mathcal{A}_1$ over a commutative semiring S, there exists an ε-free two-mode alternating weighted automaton $\mathcal{A}_2$ over S such that $|\mathcal{A}_2| = |\mathcal{A}_1|$.*

Proof. Let $\mathcal{A}_1 = (Q_1^{\oplus}, Q_1^{\otimes}, \Sigma, T_1, \nu_1, \iota_1, \tau_1)$ be a two-mode alternating weighted automaton over a commutative semiring S. We shall construct an ε-free two-mode alternating weighted automaton $\mathcal{A}_2 = (Q_2^{\oplus}, Q_2^{\otimes}, \Sigma, T_2, \nu_2, \iota_2, \tau_2)$ over S such that $|\mathcal{A}_2| = |\mathcal{A}_1|$.

Let us first describe the construction informally. In a "naive" approach, one could argue as follows. For each p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and each a in Σ, we need to modify the transitions in $T_{\mathcal{A}_1}(p, a)$ in such a way, that p "realizes" the polynomial $P_{\mathcal{A}_1}[p, a]$. The terminal weight $\tau(p)$ then needs to be changed to $(|\mathcal{A}_1|, \varepsilon)$. If we manage to do this, no ε-labelled transitions are needed in the modified $\mathcal{A}_1$. The problem with this approach is that it might not be possible to make the state p realize the polynomial $P_{\mathcal{A}_1}[p, a]$. Each state in a two-mode alternating weighted automaton can perform either addition only or multiplication only, but the polynomial $P_{\mathcal{A}_1}[p, a]$ might combine both addition and multiplication of indeterminates. This means that the polynomial $P_{\mathcal{A}_1}[p, a]$ can be realized only in two steps: in the first step, a sum state realizes the addition in $P_{\mathcal{A}_1}[p, a]$ and in the second step, a set of product states realizes the multiplication in this polynomial. The problem is that we need to "read" only one symbol during these two steps. This is impossible in an ε-free two-mode alternating weighted automaton.

To make the naive approach work, we need to modify each state p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ in such a way that it realizes the polynomial $P_{\mathcal{A}_1}[p, ab]$ in two steps for each a, b in Σ. We shall do this as follows. Each state p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ is changed to a sum state and the transitions in $T_{\mathcal{A}_1}(p, a)$ are changed in such a way that they realize the addition in $P_{\mathcal{A}_1}[p, ab]$ when reading the symbol a. For each monomial m in $P_{\mathcal{A}_1}[p, ab]$, we add a product state q_m to $\mathcal{A}_1$ that realizes the monomial m on reading the symbol b. Of course, we need to deal also with the case when there is only one symbol c left on the "input". For this reason, we need to add a transition with label c and weight $(|\mathcal{A}_1|_p, c)$ to $T(p, c)$ and lead it into a new state q_d that has no transitions starting in it; if there is more than one symbol left on the input, the branch that visits this state "returns" zero.

We shall now give a formal description of the construction. For each p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and w in Σ^*, the evaluation polynomial $P_{\mathcal{A}_1}[p, w]$ can be written as a sum of distinct nonzero monomials $m_1, m_2, \ldots, m_k$ in $S[x_1, \ldots, x_n]$ (where $n = |Q_1^{\oplus} \cup Q_1^{\otimes}|$). Let $M(p, w) = \{m_i \mid i = 1, \ldots, k\}$.

First, let us take $Q_2^{\oplus} = Q_1^{\oplus} \cup Q_1^{\otimes} \cup \{q_d\}$, where q_d is a new state that is not in $Q_1^{\oplus} \cup Q_1^{\otimes}$, and let $Q_2^{\otimes}$ consist of all pairs (m, b), where b is in Σ and m is in $M(p, ab)$ for some p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and a in Σ. Let the states of $\mathcal{A}_2$ be somehow linearly ordered – we require only that each state in $Q_1^{\oplus} \cup Q_1^{\otimes}$ keeps its numerical order from the ordering of states of $\mathcal{A}_1$.

The set of transitions T_2 will be constructed from the following sets:

- T_α that consists of all tuples (p, a, s, q_d), where p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, a is in Σ, and $s = (|\mathcal{A}_1|_p, a)$.
- T_β that consists of all tuples $(p, a, s, (m, b))$, where p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, a, b are in Σ, m is a monomial in $M(p, ab)$, and $s = coef(m)$.
- T_γ that consists of all tuples $((m, a), a, 1_S, p, i)$, where m is in $S[x_1, ..., x_n]$, a is in Σ, (m, a) is in $Q_2^{\otimes}$, p is the j-th state in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and i is a positive integer such that $i \le exp(m, j)$.

We now define $T_2 = T_\alpha \cup T_\beta \cup T_\gamma$. For every transition t in T_2, we define $init(t)$, $\sigma(t)$, $\nu(t)$ and $ter(t)$ to be the first, the second, the third, and the fourth entry of t, respectively.

Finally, the initial and the terminal weighting functions are defined by

$$\iota_2(p) := \begin{cases} \iota_1(p) & \text{if } p \in Q_1^{\oplus} \cup Q_1^{\otimes}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_2(p) := \begin{cases} (|\mathcal{A}_1|_p, \varepsilon) & \text{if } p \in Q_1^{\oplus} \cup Q_1^{\otimes}, \\ 1 & \text{if } p = q_d, \\ 0 & \text{otherwise.} \end{cases}$$

We shall now prove that $|\mathcal{A}_2| = |\mathcal{A}_1|$. First, we shall show that $(|\mathcal{A}_2|_p, w) = (|\mathcal{A}_1|_p, w)$ for every p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and w in Σ^*. The proof will be done by induction on the length of the word w.

Assume that $w = \varepsilon$. Since $\mathcal{A}_2$ is ε-free, we have $(|\mathcal{A}_2|_p, \varepsilon) = \tau_2(p) = (|\mathcal{A}_1|_p, \varepsilon)$. Let us now assume that $w = a$, where a is in Σ. Since $\mathcal{A}_2$ is ε-free, we have

$$(|\mathcal{A}_2|_p, a) = \sum_{t \in T_{\mathcal{A}_2}(p,a)} \nu_2(t) \cdot \tau_2(ter(t)).$$

There is a single transition t' in $T_{\mathcal{A}_2}(p, a)$ that ends in state q_d and this transition satisfies $\nu_2(t') = (|\mathcal{A}_1|_p, a)$ and $\tau_2(ter(t')) = 1$. Every other transition t in $T_{\mathcal{A}_2}(p, a)$ ends in $Q_2^{\otimes}$ and hence $\tau_2(ter(t)) = 0$ by the definition of the terminal weighting function τ_2. It follows that

$$(|\mathcal{A}_2|_p, a) = \nu_2(t') \cdot \tau_2(ter(t')) = (|\mathcal{A}_1|_p, a).$$

Finally, let us assume that $w = abv$ for some a, b in Σ and v in Σ^*. The reader can easily check that $P_{\mathcal{A}_2}[p, ab] = P_{\mathcal{A}_1}[p, ab]$ (here we use the fact that for each state q in $Q_1^{\oplus} \cup Q_1^{\otimes}$, the numerical order of q in $\mathcal{A}_1$ and $\mathcal{A}_2$ is the same). By the induction hypothesis, we have $(|\mathcal{A}_2|_q, v) = (|\mathcal{A}_1|_q, v)$ for every q in $Q_1^{\oplus} \cup Q_1^{\otimes}$. If we put all these facts together and use Lemma 2.4.6, we obtain

$$\begin{aligned} (|\mathcal{A}_2|_p, w) &= P_{\mathcal{A}_2}[p, ab]((|\mathcal{A}_2|_1, v), (|\mathcal{A}_2|_2, v), \ldots, (|\mathcal{A}_2|_m, v)) = \\ &= P_{\mathcal{A}_1}[p, ab]((|\mathcal{A}_1|_1, v), (|\mathcal{A}_1|_2, v), \ldots, (|\mathcal{A}_1|_n, v)) = \\ &= (|\mathcal{A}_1|_p, w), \end{aligned}$$

where $m = |Q_2^{\oplus} \cup Q_2^{\otimes}|$ and $n = |Q_1^{\oplus} \cup Q_1^{\otimes}|$.

It is now easy to show that $|\mathcal{A}_2| = |\mathcal{A}_1|$. For every w in Σ^*, we have

$$(|\mathcal{A}_2|, w) = \sum_{p \in Q_2^{\oplus} \cup Q_2^{\otimes}} \iota_2(p) \cdot (|\mathcal{A}_2|_p, w).$$

If p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, then $\iota_2(p) = \iota_1(p)$ and $(|\mathcal{A}_2|_p, w) = (|\mathcal{A}_1|_p, w)$, as we have already proven. For every other state p in $Q_2^{\oplus} \cup Q_2^{\otimes}$, we have $\iota_2(p) = 0$. These facts imply

$$(|\mathcal{A}_2|, w) = \sum_{p \in Q_1^{\oplus} \cup Q_1^{\otimes}} \iota_1(p) \cdot (|\mathcal{A}_1|_p, w) = (|\mathcal{A}_1|, w).$$

The theorem is proven. □

2.6 Systems of H-Polynomial Equations

In this section, we introduce the notion of systems of H-polynomial[3] equations. We shall use this notion to give another characterization of formal power series realized by alternating weighted automata.

[3] An abbreviation for Hadamard-polynomials.

Definition 2.6.1. A *system of H-polynomial equations* $\mathscr{P}$ over a commutative semiring S and over an alphabet Σ in indeterminates $X_1, X_2, \ldots, X_n$ is a system of n equations

$$X_i = \sum_{j=1}^{l_i} \Big(s_{i,j} a_{i,j} \cdot \Big(\bigodot_{k=1}^{n} X_k^{\odot\, m_{i,j,k}} \Big) \Big) + t_i \varepsilon, \qquad i = 1, \ldots, n, \tag{2.2}$$

where l_i is a nonnegative integer, $s_{i,j}$ and t_i are in S and $a_{i,j}$ is in Σ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$ (so $s_{i,j} a_j$ and $t_i \varepsilon$ are power series in $S\langle\langle \Sigma^* \rangle\rangle$), $m_{i,j,k}$ is a nonnegative integer for $i = 1, \ldots, n$, $j = 1, \ldots, l_i$, and $k = 1, \ldots, n$, and $\sum_{k=1}^{n} m_{i,j,k} > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$.

A n-tuple $(r_1, r_2, \ldots, r_n)$ of power series in $S\langle\langle \Sigma^* \rangle\rangle$ is a *solution* to $\mathscr{P}$ if

$$r_i = \sum_{j=1}^{l_i} \Big(s_{i,j} a_{i,j} \cdot \Big(\bigodot_{k=1}^{n} r_k^{\odot\, m_{i,j,k}} \Big) \Big) + t_i \varepsilon$$

for every $i = 1, \ldots, n$.

Proposition 2.6.2. *Every system $\mathscr{P}$ of H-polynomial equations over a commutative semiring S and over an alphabet Σ has exactly one solution.*

Proof. Let $\mathscr{P}$ consist of n equations of the form (2.2). For $i = 1, \ldots, n$, we shall inductively define a formal power series r_i in $S\langle\langle \Sigma^* \rangle\rangle$. Let the coefficient of ε in r_i be t_i. For every a in Σ, let $J_{a,i}$ be the set of indices j, for which $a_{i,j} = a$, and let us define

$$(r_i, aw) = \sum_{j \in J_{a,i}} \Big(s_{i,j} \cdot \prod_{k=1}^{n} (r_k, w)^{m_{i,j,k}} \Big)$$

for each w in Σ^*. These relations define the formal power series r_i for $i = 1, \ldots, n$. Clearly, the n-tuple $(r_1, r_2, \ldots, r_n)$ is a solution to $\mathscr{P}$. Moreover, this n-tuple is evidently the only possible solution to $\mathscr{P}$. □

Let $\mathscr{P}$ be a system of n H-polynomial equations and $(r_1, r_2, \ldots, r_n)$ be the solution to $\mathscr{P}$. We shall write $|\mathscr{P}|_i$ to denote r_i for $i = 1, \ldots, n$. We claim that a power series r over a commutative semiring S and over an alphabet Σ is realized by an alternating weighted automaton over S iff there exists a system $\mathscr{P}$ of H-polynomial equations over S and Σ such that $|\mathscr{P}|_1 = r$. We start the proof of this claim with the following lemma.

Lemma 2.6.3. *For every alternating weighted automaton $\mathcal{A}$ over a commutative semiring S, there exists an alternating weighted automaton $\mathcal{A}' = (Q, \Sigma, \psi, P_0, \tau)$ over S such that $P_0 = x_1$ and $|\mathcal{A}'| = |\mathcal{A}|$.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma, \psi_1, P_{0,1}, \tau_1)$ be an alternating weighted automaton over S, let $n = |Q_1|$. We shall construct an alternating weighted automaton $\mathcal{A}_2 = (Q_2, \Sigma, \psi_2, P_{0,2}, \tau_2)$ with $n+1$ states such that $P_{0,2} = x_{n+1}$ and $|\mathcal{A}_2| = |\mathcal{A}_1|$. Once this is done, the initial polynomial $P_{0,2}$ can be changed to x_1 if the states of $\mathcal{A}_2$ are suitably rearranged.

Let us define $Q_2 = Q_1 \cup \{q_0\}$, where q_0 is a new state that is not in Q_1. The numerical order of q_0 is $n+1$, while the rest of states in Q_2 keeps its ordering from Q_1. For every a in Σ, let us define $\psi_2[q_0, a] = P_{0,1}(\psi_1[1, a], \psi_1[2, a], \ldots, \psi_1[n, a])$. The terminal weight of q_0 is defined by $\tau_2(q_0) = P_{0,1}(\tau_1(1), \tau_1(2), \ldots, \tau_1(n))$. For every p in $Q_2 - \{q_0\}$ and a is in Σ, the polynomial $\psi_2[p, a]$ and the terminal weight $\tau_2(p)$ does not change, i.e., $\psi_2[p, a] = \psi_1[p, a]$ and $\tau_2(p) = \tau_1(p)$. Finally, let $P_{0,2} = x_{n+1}$. It is easy to show that $|\mathcal{A}_2| = |\mathcal{A}_1|$. □

The following theorem gives a characterization of formal power series realized by alternating weighted automata in terms of systems of H-polynomial equations.

Theorem 2.6.4. *Let r be a formal power series over a commutative semiring S and over an alphabet Σ. There exists an alternating weighted automaton $\mathcal{A}$ over S such that $|\mathcal{A}| = r$ iff there exists a system of H-polynomial equations $\mathscr{P}$ over S such that $|\mathscr{P}|_1 = r$.*

Proof. Let $\mathscr{P}$ be a system of H-polynomial equations over S and Σ that consist of n equations of the form (2.2). We shall construct an alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ such that $|\mathcal{A}| = |\mathscr{P}|_1$.

Let us define $Q = \{1, 2, \ldots, n\}$. For every i in Q and a in Σ, let $J_{a,i}$ be a set of indices j for which $a_{i,j} = a$ and let us define

$$\psi[i, a] = \sum_{j \in J_{a,i}} s_{i,j} \prod_{k=1}^{n} x_i^{m_{i,j,k}}.$$

We define $P_0 = x_1$ and $\tau(i) = t_i$ for every i in Q. It is not too hard to show that $|\mathcal{A}| = |\mathscr{P}|_1$. The proof of this fact is left for the reader.

Let us now prove the converse implication of the theorem. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over S. We shall construct a system of H-polynomial equations $\mathscr{P}$ over S and Σ such that $|\mathscr{P}|_1 = |\mathcal{A}|$.

By Lemma 2.6.3, we can assume that $P_0 = x_1$. The system of H-polynomial equations $\mathscr{P}$ shall consist of $n := |Q|$ equations. Let us construct the i-th equation of $\mathscr{P}$ for some positive integer i in $\{1, \ldots, n\}$. For every a in Σ, we can write

$$\psi[i, a] = \sum_{j=1}^{l_a} s_{a,j} \prod_{k=1}^{n} x_k^{m_{a,j,k}},$$

where l_a is a nonnegative integer, $s_{a,j}$ is in S for $j = 1, \ldots, l_a$, the exponent $m_{a,j,k}$ is a nonnegative integer for $j = 1, \ldots, l_a$ and $k = 1, \ldots, n$, and $\sum_{k=1}^{n} m_{a,j,k} > 0$ for $j = 1, \ldots, l_a$. Let the i-th equation of $\mathscr{P}$ be

$$X_i = \sum_{a \in \Sigma} \Big(\sum_{j=1}^{l_a} s_{a,j} a \cdot \Big(\bigodot_{k=1}^{n} X_k^{\odot\, m_{a,j,k}} \Big) \Big) + \tau(i)\varepsilon.$$

It is easy to show that $|\mathscr{P}|_1 = |\mathcal{A}|$. □

Chapter 3

Series Realized by Alternating Weighted Automata

3.1 Expressive Power

Since (nonalternating) weighted automata are just a special case of alternating weighted automata, the expressive power of the latter is at least as big as that of the former. A natural question is if alternating weighted automata are *strictly* more powerful. A positive answer to this question was already given by Almagor and Kupferman [1], who constructed a simple alternating weighted automaton over a tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ and showed that no (nonalternating) weighted automaton over the same semiring is equivalent to it. Therefore, alternating weighted automata over the tropical semiring are strictly more powerful than (nonalternating) weighted automata over the tropical semiring. However, not the same can be said about the Boolean semiring. It is already well known that a language accepted by a Boolean alternating automaton is necessarily regular [2]. This shows that alternating weighted automata and (nonalternating) weighted automata over the Boolean semiring are equally powerful.

We conclude that commutative semirings can be divided into two nonempty classes: the class of commutative semirings, for which alternating weighted automata and (nonalternating) weighted automata are equally powerful and the class of commutative semirings, for which alternating weighted automata are strictly more powerful than (nonalternating) weighted automata. Let $\mathscr{S}$ denote the former class, i.e., let $\mathscr{S}$ be the class that consists of all commutative semirings S such that the behaviour of every alternating weighted automaton over S is rational over S. For the rest of this section, we shall be proving the following simple characterization of the class $\mathscr{S}$.

Theorem 3.1.1. *For every commutative semiring S, the following assertions are equivalent:*

1. *The semiring S is in $\mathscr{S}$.*
2. *Every finitely generated subsemiring of S is finite.*
3. *Every element of S has finite multiplicative order.*

As a first step, we shall prove the equivalence of assertions 2 and 3. Let us state this equivalence once more in the following lemma and prove it.

Lemma 3.1.2. *For every commutative semiring S, the following two assertions are equivalent:*

1. *Every finitely generated subsemiring of S is finite.*
2. *Every element of S has finite multiplicative order.*

Proof. Let us suppose that every finitely generated subsemiring of S is finite. In particular, this implies that the subsemiring T_s generated by a semiring element s is finite for each s in S. The subsemiring T_s contains s^i for every nonnegative integer i and therefore, there is only a finite number of such elements. This means that s has finite multiplicative order.

For the converse implication, let us suppose that every element of S has finite multiplicative order. First of all, let us note that every element of S has a finite *additive* order as well. Since 2_S has finite multiplicative order, there exist two distinct nonnegative integers k_1, k_2 such that $(2_S)^{k_1} = (2_S)^{k_2}$. We have

$$2^{k_1}1_S = (2^{k_1})_S = (2_S)^{k_1} = (2_S)^{k_2} = (2^{k_2})_S = 2^{k_2}1_S,$$

which shows that there exist two distinct nonnegative integers $l_1 = 2^{k_1}$ and $l_2 = 2^{k_2}$ such that $l_1 1_S = l_2 1_S$. Therefore, if t is in S, then $l_1 t = (l_1 1_S)t = (l_2 1_S)t = l_2 t$. We have thus shown that $l_1 t = l_2 t$ for two distinct nonnegative integers l_1, l_2, which means that t has finite additive order.

Let T be a subsemiring of S generated by elements $s_1, s_2, \dots, s_n$ in S. We shall show that T is finite. Let M be a subset of S that consists of all elements $\prod_{i=1}^{n} s_i^{k_i}$, where k_i is a nonnegative integer for $i = 1, \dots, n$. Since s_i has finite multiplicative order for $i = 1, \dots, n$, the set M is finite. Let K be a subset of S that consists of all elements $\sum_{t \in M} k_t t$, where k_t is a nonnegative integer for every t in M. Since every t in M has finite additive order, we can see that K is finite as well. Moreover, it is quite easy to show that K is a subsemiring of S that contains $s_1, \dots, s_n$ and is contained in every subsemiring of S that contains $s_1, \dots, s_n$. This means that K is a subsemiring generated by $s_1, \dots, s_n$ and hence, $K = T$. This proves that T is finite. □

We shall now prove the following part of Theorem 3.1.1: if every finitely generated subsemiring of S is finite, then S is in $\mathscr{S}$. First, we shall prove this claim in the case when S is itself finitely generated and hence is finite.

Lemma 3.1.3. *If S is a finite commutative semiring, then S is in $\mathscr{S}$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over S. We shall construct a (nonalternating) weighted automaton $\mathcal{A}' = (Q', \Sigma, T, \nu, \iota, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$. This construction was already done by Chandra, Kozen and Stockmeyer [2] in the special case when S is the Boolean semiring. The construction for an arbitrary finite commutative semiring S that we present here is a generalization of the construction by Chandra, Kozen and Stockmeyer.

Let $n = |Q|$. Since S is finite, there exists a finite set $\rho_1, \rho_2, \dots, \rho_m$ of polynomial functions in $S(x_1, \dots, x_n)$ such that every polynomial function η in $S(x_1, \dots, x_n)$ is a linear combination of polynomial functions $\rho_1, \dots, \rho_m$, i.e., we can write $\eta = \sum_{i=1}^{m} c_i \rho_i$, where c_i is in S for $i = 1, \dots, m$. For $i = 1, \dots, m$, let R_i be a polynomial in $S[x_1, \dots, x_n]$ such that ρ_i is its corresponding polynomial function.

The set of states Q' shall be defined by $Q' = \{1, 2, \dots, m\}$. Let us now define the set of transitions T and the transition weighting function ν. For every k in Q' and every a in Σ, we shall construct the set of transitions $T(k, a)$ that start in k and are labelled with a. Let η be the polynomial function that corresponds to the polynomial $R_k(\psi[1, a], \psi[2, a], \dots, \psi[n, a])$. We can write $\eta = \sum_{i=1}^{m} c_i \rho_i$, where c_i is in S for $i = 1, \dots, m$. We define the set $T(k, a)$ in such a way that for every integer i, for which c_i is nonzero, there is exactly one transition in $T(k, a)$ that ends in i; the weight of this transition is c_i. In this way, we construct $T(k, a)$ for every k in Q' and a in Σ. We now define $T = \bigcup_{k \in Q', a \in \Sigma} T(k, a)$. For each transition t in T, we have also defined the weight $\nu(t)$ of t.

Let us now define the initial weighting function ι. Let ϕ_0 be the polynomial function that corresponds to the polynomial P_0. We can write $\phi_0 = \sum_{i=1}^{m} c_i \rho_i$, where c_i is in S for $i = 1, \dots, m$. For every k in Q', we define $\iota(k) = c_k$.

Finally, let us define the terminal weighting function τ'. For every k in Q', we define $\tau'(k) = \rho_k(\tau(1), \tau(2), \dots, \tau(n))$.

Let us now show that $|\mathcal{A}'| = |\mathcal{A}|$. First, we shall prove that

$$(|\mathcal{A}'|_k, w) = \rho_k\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \dots, (|\mathcal{A}|_n, w)\big) \tag{3.1}$$

for every k in Q'. The proof will be done by mathematical induction on the length of the word w.

Let $w = \varepsilon$. We have $(|\mathcal{A}|_i, \varepsilon) = \tau(i)$ for $i = 1, \dots, n$. Since $\mathcal{A}'$ is ε-free, we have $(|\mathcal{A}'|_k, \varepsilon) = \tau'(k)$ and

$$\begin{aligned}(|\mathcal{A}'|_k, \varepsilon) &= \tau'(k) = \\ &= \rho_k(\tau(1), \tau(2), \dots, \tau(n)) = \\ &= \rho_k\big((|\mathcal{A}|_1, \varepsilon), (|\mathcal{A}|_2, \varepsilon), \dots, (|\mathcal{A}|_n, \varepsilon)\big).\end{aligned}$$

We have thus proved (3.1) for $w = \varepsilon$.

Let $w = av$, where a is in Σ and v is in Σ^*. Let η be the polynomial function that corresponds to the polynomial $R_k(\psi[1,a], \psi[2,a], \dots, \psi[n,a])$. For every i in Q', let c_i be the weight of the transition that starts at k, ends in i, and is labelled with a. If there is no such transition, let $c_i = 0$. By the definition of the set of transitions T, we have $\eta = \sum_{i=1}^m c_i \rho_i$. Since $\mathcal{A}'$ is ε-free, we also have $(|\mathcal{A}'|_k, av) = \sum_{i=1}^m c_i(|\mathcal{A}'|_i, v)$. Moreover, by the induction hypothesis, $(|\mathcal{A}'|_i, v) = \rho_i\big((|\mathcal{A}|_1, v), \dots, (|\mathcal{A}|_n, v)\big)$ for $i = 1, \dots, m$. These facts imply

$$\begin{aligned}(|\mathcal{A}'|_k, av) &= \sum_{i=1}^m c_i(|\mathcal{A}'|_i, v) = \\ &= \sum_{i=1}^m c_i\rho_i\big((|\mathcal{A}|_1, v), (|\mathcal{A}|_2, v), \dots, (|\mathcal{A}|_n, v)\big) = \\ &= \eta\big((|\mathcal{A}|_1, v), (|\mathcal{A}|_2, v), \dots, (|\mathcal{A}|_n, v)\big) = \\ &= R_k\Big(\psi[1,a]\big((|\mathcal{A}|_1, v), (|\mathcal{A}|_2, v), \dots, (|\mathcal{A}|_n, v)\big), \\ &\qquad \psi[2,a]\big((|\mathcal{A}|_1, v), (|\mathcal{A}|_2, v), \dots, (|\mathcal{A}|_n, v)\big), \\ &\qquad \dots, \\ &\qquad \psi[n,a]\big((|\mathcal{A}|_1, v), (|\mathcal{A}|_2, v), \dots, (|\mathcal{A}|_n, v)\big)\Big) = \\ &= \rho_k\big((|\mathcal{A}|_1, av), (|\mathcal{A}|_2, av), \dots, (|\mathcal{A}|_n, av)\big).\end{aligned}$$

We have thus proved (3.1) for $w = av$.

We shall now prove that $(|\mathcal{A}'|, w) = (|\mathcal{A}|, w)$ for every w in Σ^*. Let ϕ_0 be the polynomial function that corresponds to the polynomial P_0. By the definition of the initial weighting function ι, we have $\phi_0 = \sum_{i=1}^m \iota(i)\rho_i$. This fact and (3.1) imply

$$\begin{aligned}(|\mathcal{A}'|, w) &= \sum_{i=1}^m \iota(i)(|\mathcal{A}'|_i, w) = \\ &= \sum_{i=1}^m \iota(i)\rho_i\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \dots, (|\mathcal{A}|_n, w)\big) = \\ &= \phi_0\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \dots, (|\mathcal{A}|_n, w)\big) = \\ &= (|\mathcal{A}|, w).\end{aligned}$$

The lemma is proven. □

We now wish to prove the more general claim: if every finitely generated subsemiring of S is finite, then S is in $\mathscr{S}$, regardless of whether the semiring S is finite or not. This claim follows easily from the special case when S is finite. The reason is that even if S is not finite, every alternating weighted automaton over S still "makes use" of only some finite part of it. Let us suppose that $\mathcal{A}$ is some two-mode alternating weighted automaton over S. If X is the set of all elements of S that are either carried by some transition of $\mathcal{A}$ or are assigned to some state as an initial or a terminal weight, then $\mathcal{A}$ can be viewed as a two-mode alternating weighted automaton over the semiring T that is generated by the elements of X. The set X is clearly finite, so the semiring T is finitely generated. The same reasoning stands also if $\mathcal{A}$ is a (general) alternating weighted automaton. We shall now summarize these observations in the following lemma.

Lemma 3.1.4. *For every alternating weighted automaton $\mathcal{A}$ over a commutative semiring S, there exists a finitely generated subsemiring T of S and an alternating weighted automaton $\mathcal{A}'$ over T such that $|\mathcal{A}'| = |\mathcal{A}|$.*

We are now prepared to give a proof of the following part of Theorem 3.1.1.

Lemma 3.1.5. *If S is a commutative semiring such that every finitely generated subsemiring of S is finite, then S is in $\mathscr{S}$.*

Proof. Let $\mathcal{A}$ be an alternating weighted automaton over S. We shall prove that there exists a (nonalternating) weighted automaton $\mathcal{B}$ over S such that $|\mathcal{B}| = |\mathcal{A}|$. By Lemma 3.1.4, there exists a finitely generated subsemiring T of S and an alternating weighted automaton $\mathcal{A}'$ over T such that $|\mathcal{A}'| = |\mathcal{A}|$. The semiring T is finite and so it is in $\mathscr{S}$ by Lemma 3.1.3. Therefore, there exists a (nonalternating) weighted automaton $\mathcal{B}$ over T such that $|\mathcal{B}| = |\mathcal{A}'| = |\mathcal{A}|$. Clearly, $\mathcal{B}$ can be viewed as a (nonalternating) weighted automaton over the semiring S. □

To finish the proof of Theorem 3.1.1, we shall prove the following: if a commutative semiring S is in $\mathscr{S}$, then every element of S has finite multiplicative order. This is the most difficult part of our proof of Theorem 3.1.1 and requires some preparations.

For every nonnegative integer n, let $\langle n\rangle$ denote the binary representation of n ($\langle n\rangle$ is a word over the alphabet $\{0,1\}$). Let $X = \{s_1, s_2, \ldots, s_z\}$ be a finite subset of a commutative semiring S. It is not too hard to construct an alternating weighted automaton $\mathcal{A}$ over S and over the alphabet $\{0, 1, \#\}$ such that

$$(|\mathcal{A}|, \langle k_1\rangle\#\langle k_2\rangle\#\ldots\#\langle k_z\rangle) = \prod_{i=1}^{z} s_i^{k_i} \tag{3.2}$$

for every z-tuple of nonnegative integers $k_1, k_2, \ldots, k_z$.[1] A diagram of one such automaton for case $z = 3$ is depicted in Figure 3.1. For every finite subset $X = \{s_1, \ldots, s_z\}$ of S, let r_X be a formal power series over S and over $\{0, 1, \#\}$ that is realized by an alternating weighted automaton over S and satisfies (3.2) for every z-tuple of nonnegative integers $k_1, \ldots, k_z$. We claim that if the formal power series r_X is rational over S for every finite subset X of S, then every element of S has finite multiplicative order. We shall now give some definitions that will help us to prove this.

For every finite subset $X = \{s_1, s_2, \ldots, s_z\}$ of S and every nonnegative real number c, let $G_X(c)$ be the set that consists of all elements s in S such that

$$s = \prod_{i=1}^{z} s_i^{k_i}$$

for some nonnegative integers $k_1, k_2, \ldots, k_z$ satisfying $\sum_{i=1}^{z} k_i \le c$. Furthermore, for every pair of nonnegative real numbers c, d, let $H_X(c, d)$ be the set that consists of all elements s in S such that

$$s = \sum_{i=1}^{k} g_i,$$

where k is a nonnegative integer satisfying $k \le d$ and g_i is in $G_X(c)$ for $i = 1, \ldots, k$. If X contains only one element s, we can write $G_s(c)$ and $H_s(c, d)$ instead of $G_{\{s\}}(c)$ and $H_{\{s\}}(c, d)$.

The sets $G_X(c)$ and $H_X(c, d)$ are clearly finite for every finite subset X of S and for every pair of nonnegative real numbers c, d. Is is easy to see that if X consists of z elements, then the sizes

[1]Note that (3.2) does not specify value of r_X on each word in $\{0, 1, \#\}^*$, but this is not important for our purposes.

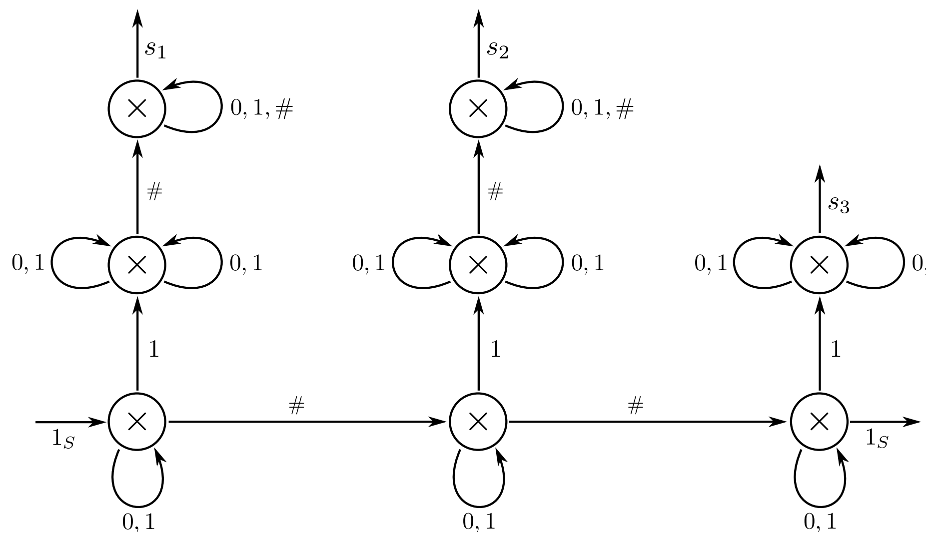

Figure 3.1: A two-mode alternating weighted automaton $\mathcal{A}$ over the alphabet $\{0, 1, \#\}$ and over a commutative semiring S that contains some particular elements s_1, s_2, s_3. The formal power series $|\mathcal{A}|$ satisfies $(|\mathcal{A}|, \langle k_1\rangle\#\langle k_2\rangle\#\langle k_3\rangle) = s_1^{k_1} s_2^{k_2} s_3^{k_3}$ for every triplet of nonnegative integers k_1, k_2, k_3.

of these sets can be estimated by

$$|G_X(c)| \le \binom{\lfloor c\rfloor + z}{z} \le (c+z)^z,$$
$$|H_X(c,d)| \le \binom{\lfloor d\rfloor + |G_X(c)|}{|G_X(c)|} \le (d + |G_X(c)|)^{|G_X(c)|} \le (d + (c+z)^z)^{(c+z)^z}.$$

For our purposes, we shall manage with the weaker estimates

$$|G_X(c)| \le (c+z)^z, \tag{3.3}$$
$$|H_X(c,d)| \le (d + (c+z)^z)^{(c+z)^z}. \tag{3.4}$$

Let us remind that our goal is to prove the following: if S is a commutative semiring in $\mathscr{S}$, then every element of S has finite multiplicative order. In what follows, we shall state and prove three lemmas that we need for our proof of this fact.

Lemma 3.1.6. *Let S be a commutative semiring and Σ be an alphabet. If r in $S\langle\langle\Sigma^*\rangle\rangle$ is rational over S, then there exists a finite subset Y of S and a real number c such that (r, w) is in $H_Y(|w|+1, c^{|w|})$ for every w in Σ^*.*

Proof. Let r be realized by a weighted automaton $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ over S. We can assume that $\mathcal{A}$ is ε-free and that $\iota(p) = 0$ for every p in Q except for some q_0 in Q for which $\iota(q_0) = 1$. It is a well known fact that every rational power series is realized by some weighted automaton that satisfies these conditions [5]. Clearly, we can also assume that there are no parallel transitions with the same label in $\mathcal{A}$.

Let Y be the set that consists of all elements of S that are carried by some transition in T or are assigned to some state in Q as a terminal weight, i.e.,

$$Y = \{\nu(t) \mid t \in T\} \cup \{\tau(p) \mid p \in Q\}.$$

Let $c = |Q|$. We claim that for every p in Q and w in Σ^*, the coefficient of w in $|\mathcal{A}|_p$ is in $H_Y(|w|+1, c^{|w|})$. Once we prove this claim, the proof of Lemma 3.1.6 is finished, because $|\mathcal{A}| = |\mathcal{A}|_{q_0}$. We shall prove the claim by induction on the length of w.

If $w = \varepsilon$, then $(|\mathcal{A}|_p, w) = \tau(p)$. Since $\tau(p)$ is in Y, the coefficient of w in $|\mathcal{A}|_p$ is in $H_Y(1,1) = H_Y(|w|+1, c^{|w|})$. Let us now assume that $w = av$ for some a in Σ and v in Σ^*. We have

$$(|\mathcal{A}|_p, av) = \sum_{t \in T(p,a)} \nu(t) \cdot (|\mathcal{A}|_{ter(t)}, v). \tag{3.5}$$

By the induction hypothesis, $(|\mathcal{A}|_{ter(t)}, v)$ is in $H_Y(|v|+1, c^{|v|})$ for every t in $T(p,a)$ and so $\nu(t) \cdot (|\mathcal{A}|_{ter(t)}, v)$ is in $H_Y(|v|+2, c^{|v|})$ for every t in $T(p,a)$. Moreover, there are no more than $c = |Q|$ transitions in $T(p,a)$, since $\mathcal{A}$ has no parallel transitions with the same label. So the sum in (3.5) is taken over at most c elements from $H_Y(|v|+2, c^{|v|})$ and hence, $(|\mathcal{A}|_p, w)$ is in $H_Y(|v|+2, c^{|v|+1}) = H_Y(|w|+1, c^{|w|})$. □

Lemma 3.1.7. *Let S be a commutative semiring in $\mathscr{S}$. For every finite subset X of S, there exists a nonnegative integer n_1, a real number d_1 and a finite subset Y of S such that $G_X(n) \subseteq H_Y(d_1 \log n, n^{d_1})$ for every $n \geq n_1$*

Proof. Let $X = \{s_1, s_2, \ldots, s_z\}$. Since S is in $\mathscr{S}$, the series r_X is rational over S. Let Y be a finite subset of S and c be a real number such that $c \geq 1$ and

$$(r_X, w) \in H_Y(|w|+1, c^{|w|}) \tag{3.6}$$

for every w over the alphabet $\{0, 1, \#\}$. Existence of such Y and c is guaranteed by Lemma 3.1.6. Let n_1 be an integer and d_1 be a real number such that

$$z(\log n + 2) + 1 \leq d_1 \log n \quad \text{and} \quad c^{2z} n^{z \log c} \leq n^{d_1} \tag{3.7}$$

holds for every $n \geq n_1$.

Let n be an integer satisfying $n \geq n_1$, let s be in $G_X(n)$. Our goal is to prove that s is in $H_Y(d_1 \log n, n^{d_1})$. We can write

$$s = \prod_{i=1}^{z} s_i^{k_i},$$

where $\sum_{i=1}^{z} k_i \leq n$. The element s is also the coefficient of the word $w := \langle k_1 \rangle \# \langle k_2 \rangle \# \ldots \# \langle k_z \rangle$ in r_X. The reader can easily check that $|w| \leq z(\log n + 2)$. By (3.6), the element s is in $H_Y(z(\log n + 2) + 1, c^{z(\log n + 2)}) = H_Y(z(\log n + 2) + 1, c^{2z} n^{z \log c})$. By (3.7), this set is included in $H_Y(d_1 \log n, n^{d_1})$. So s is in $H_Y(d_1 \log n, n^{d_1})$ and the lemma is proven. □

Lemma 3.1.8. *Let S be commutative semiring in $\mathscr{S}$. For every s in S, there exists a nonnegative integer n_0, real numbers c_1, c_2, and a finite subset Y of S such that $G_s(2^n)$ is included in $H_Y(c_1 \log n, n^{c_2})$ for every $n \geq n_0$.*

Proof. Let n_1 be an integer, d_1 be a real number, and X be a finite subset of S such that

$$G_s(n) \subseteq H_X(d_1 \log n, n^{d_1}) \tag{3.8}$$

holds for every $n \geq n_1$. Existence of such n_1, d_1, and X is guaranteed by Lemma 3.1.7. Assume that $d_1 \geq 1$. If X does not contain 2_S, let us add this semiring element to X (this has no effect on the inclusion (3.8)). Similarly, let n_2 be an integer, d_2 be a real number, and Y be a finite subset of S such that

$$G_X(n) \subseteq H_Y(d_2 \log n, n^{d_2}) \tag{3.9}$$

holds for every $n \geq n_2$. Let us denote $z = |Y|$. Let n_3 be an integer and c_1 be a real number such that

$$d_2 \log(2 d_1 n) \leq c_1 \log n, \tag{3.10}$$

holds for every $n \geq n_3$. Let n_4 be an integer and c_2 be a real number such that

$$\tfrac{1}{2}(2 d_1 n)^{d_2 + 1} (d_1 n + z)^z \leq n^{c_2} \tag{3.11}$$

holds for every $n \geq n_4$. Finally, let $n_0 = \max\{n_1, n_2, n_3, n_4\}$.

Let n be an integer satisfying $n \geq n_0$, let t be in $G_s(2^n)$. Our goal is to prove that t is in $H_Y(c_1 \log n, n^{c_2})$. By (3.8), t is in $H_X(d_1 \log 2^n, (2^n)^{d_1}) = H_X(d_1 n, 2^{d_1 n})$. This means that we can write

$$t = \sum_{g \in G_X(d_1 n)} k_g g, \tag{3.12}$$

where k_g is a nonnegative integer satisfying

$$k_g \leq 2^{d_1 n} \tag{3.13}$$

for every g in $G_X(d_1 n)$. Let us pick some g from $G_X(d_1 n)$ and examine the semiring element $h := k_g g$. If we take into account the binary representation of the integer k_g, we can see that

$$k_g = \sum_{i=0}^{l} 2^{m_i}$$

for some $l \leq \log(k_g)$ and $m_i \leq \log(k_g)$ for $i = 0, 1, \ldots, l$. We thus have

$$h = \sum_{i=0}^{l} h_i. \tag{3.14}$$

where $h_i := (2_S)^{m_i} g$ for $i = 0, 1, \ldots, l$. Since g is in $G_X(d_1 n)$, the element h_i is in $G_X(m_i + d_1 n)$ for $i = 0, 1, \ldots, l$. The set $G_X(m_i + d_1 n)$ is included in $G_X(2d_1 n)$, since $m_i \leq \log k_g \leq \log 2^{d_1 n} = d_1 n$ (the second inequality follows from (3.13)). Furthermore, the inclusion (3.9) implies that $G_X(2d_1 n)$ is included in $H_Y(d_2 \log(2d_1 n), (2d_1 n)^{d_2})$. We conclude that h_i is in $H_Y(d_2 \log(2d_1 n), (2d_1 n)^{d_2})$ for $i = 0, 1, \ldots, l$. This fact, together with (3.14), implies that h is in $H_Y(d_2 \log(2d_1 n), (2d_1 n)^{d_2} l)$. Since $l \leq \log(k_g) \leq \log 2^{d_1 n} = d_1 n$ (the second inequality follows from (3.13)), this set is a subset of $H_Y(d_2 \log(2d_1 n), (2d_1 n)^{d_2} d_1 n) = H_Y(d_2 \log(2d_1 n), \frac{1}{2}(2d_1 n)^{d_2+1})$. Therefore, h is in $H_Y(d_2 \log(2d_1 n), \frac{1}{2}(2d_1 n)^{d_2+1})$. The element h was chosen as $k_g g$ for arbitrary g in $G_X(d_1 n)$, so we conclude that

$$k_g g \in H_Y(d_2 \log(2d_1 n), \tfrac{1}{2}(2d_1 n)^{d_2+1}) \tag{3.15}$$

for every g in $G_X(d_1 n)$.

Finally, let us return to equality (3.12). By (3.3), $G_X(d_1 n)$ contains no more than $(d_1 n + z)^z$ elements. Together with (3.15), this upper bound for the size of $G_X(d_1 n)$ implies that t is in $H_Y(d_2 \log(2d_1 n), \frac{1}{2}(2d_1 n)^{d_2+1}(d_1 n + z)^z)$. By inequalities (3.10) and (3.11), this set is included in $H_Y(c_1 \log n, n^{c_2})$. We have thus shown that t is in $H_Y(c_1 \log n, n^{c_2})$ and the proof of Lemma 3.1.8 is complete. □

In the following lemma, we prove the remaining part of Theorem 3.1.1.

Lemma 3.1.9. *If a commutative semiring S is in $\mathscr{S}$, then every element of S has finite multiplicative order.*

Proof. Let s be in S. We shall prove that s has finite multiplicative order. Let n_0 be an integer, c_1, c_2 be real numbers and Y be a finite subset of S such that $G_s(2^n)$ is included in $H_Y(c_1 \log n, n^{c_2})$ for every $n \geq n_0$. We have just proven in Lemma 3.1.8 that this assumption is valid. Let $z = |Y|$. By (3.4), the size of $H_Y(c_1 \log n, n^{c_2})$ is at most

$$((c_1 \log n + z)^z + n^{c_2})^{(c_1 \log n + z)^z} = 2^{\log((c_1 \log n + z)^z + n^{c_2})(c_1 \log n + z)^z}.$$

Clearly, there exists an integer n_1 such that $|H_Y(c_1 \log n, n^{c_2})| < 2^n$ for every $n \geq n_1$. If $n = \max\{n_0, n_1\}$ then $G_s(2^n)$ is a subset of $H_Y(c_1 \log n, n^{c_2})$ and therefore also $|G_s(2^n)| < 2^n$. This implies, that $s^{m_1} = s^{m_2}$ for two distinct nonnegative integers m_1, m_2, which means that s has finite multiplicative order. □

The proof of Theorem 3.1.1 is now at its end. Let us state the Theorem 3.1.1 once more.

Theorem 3.1.1. *For every commutative semiring S, the following assertions are equivalent:*

1. *The semiring S is in $\mathscr{S}$.*
2. *Every finitely generated subsemiring of S is finite.*
3. *Every element of S has finite multiplicative order.*

Proof. The equivalence of assertions 2 and 3 is guaranteed by Lemma 3.1.2. By Lemma 3.1.5, the assertion 2 implies the assertion 1. Finally, Lemma 3.1.9 shows that assertion 1 implies the assertion 3. □

Now that we have proved Theorem 3.1.1, we can use it to examine the expressive power of alternating weighted automata over some particular commutative semirings. The class of commutative semirings, for which weighted automata and alternating weighted automata are equally powerful, includes the following semirings:

- All finite commutative semirings, e.g., the Boolean semiring $(\mathbb{B}, \vee, \wedge, 0, 1)$, the semiring $(\mathbb{Z}_k, +, \cdot, 0, 1)$ of integers modulo k (for some $k \geq 2$) with standard operations of addition and multiplication, etc.
- The semiring $(\mathscr{P}(U), \cup, \cap, \emptyset, U)$ on the powerset $\mathscr{P}(U)$ of an arbitrary set U with union as addition and intersection as multiplication.

On the contrary, we shall now list some commutative semirings, for which alternating weighted automata are strictly more powerful than (nonalternating) weighted automata:

- The semiring $(\mathbb{R}_+, +, \cdot, 0, 1)$ of nonnegative real numbers with the standard operations of addition and multiplication.
- The tropical semiring, i.e., the semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ on the set of real numbers with positive infinity, together with minimum as addition and the standard addition of real numbers as multiplication.
- The arctic semiring, i.e., the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ on the set of real numbers with negative infinity, together with maximum as addition and the standard addition of real numbers as multiplication.
- The semiring of polynomials $S[x_1, \ldots, x_n]$ for an arbitrary commutative semiring S and positive integer n.
- The semiring $(2^{\{a\}^*}, \cup, \cdot, \emptyset, \{\varepsilon\})$ of formal languages over a singleton alphabet $\{a\}$, together with union as addition and concatenation as multiplication.

3.2 Closure Properties

For any individual commutative semiring S, one can examine the closure properties of the class of formal power series realized by alternating weighted automata over S. Naturally, the closure properties of these classes might vary for different semirings. In this section, we shall examine several standard operations on formal power series and for each one of them, we shall determine if the following is true: for every commutative semiring S, the class of formal power series realized by alternating weighted automata over S is closed under the operation in consideration.

We start with the examination of sum and Hadamard product of formal power series.

Theorem 3.2.1. *For every commutative semiring S, the class of formal power series realized by alternating weighted automata over S is closed under sum.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \psi_1, P_{0,1}, \tau_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma_2, \psi_2, P_{0,2}, \tau_2)$ be alternating weighted automata over S with $n_1 := |Q_1|$ and $n_2 := |Q_2|$. Let us assume that $Q_1 \cap Q_2 = \emptyset$. We shall construct an alternating weighted automaton $\mathcal{A}_3 = (Q_3, \Sigma_1 \cup \Sigma_2, \psi_3, P_{0,3}, \tau_3)$ over S such that $|\mathcal{A}_3| = |\mathcal{A}_1| + |\mathcal{A}_2|$.

First, we define $Q_3 = Q_1 \cup Q_2$. If p is the i-th state of $\mathcal{A}_1$, we choose it to be the i-th state in $\mathcal{A}_3$. If p is the i-th state of $\mathcal{A}_2$, we choose it to take the $(i + n_1)$-th position in the ordering of states of $\mathcal{A}_3$.

We shall now define the polynomial assigning function ψ_3. If P is a polynomial in $S[x_1, \ldots, x_{n_2}]$, we shall write $\mathit{shift}(P)$ to denote the polynomial in $S[x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2}]$ that is obtained from P if each occurrence of the indeterminate x_i in P is replaced by the indeterminate x_{i+n_1} for $i = 1, \ldots, n_2$. If p is in Q_1, we define $\psi_3[p, a] = \psi_1[p, a]$ for each a in Σ_1 and $\psi_3[p, a] = 0$ for each a in $\Sigma_2 - \Sigma_1$. If p is in Q_2, we define $\psi_3[p, a] = \mathit{shift}(\psi_2[p, a])$ for each a in Σ_2 and $\psi_3[p, a] = 0$ for each a in $\Sigma_1 - \Sigma_2$.

Finally, we define $P_{0,3} = P_{0,1} + \mathit{shift}(P_{0,2})$. Every state in Q_3 keeps its terminal weight from $\mathcal{A}_1$ or $\mathcal{A}_2$. It is easy to show that $|\mathcal{A}_3| = |\mathcal{A}_1| + |\mathcal{A}_2|$. □

Theorem 3.2.2. *For every commutative semiring S, the class of formal power series realized by alternating weighted automata over S is closed under Hadamard product.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \psi_1, P_{0,1}, \tau_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma_2, \psi_2, P_{0,2}, \tau_2)$ be alternating weighted automata over S with $n_1 := |Q_1|$ and $n_2 := |Q_2|$. We shall construct an alternating weighted automaton $\mathcal{A}_3 = (Q_3, \Sigma_1 \cup \Sigma_2, \psi_3, P_{0,3}, \tau_3)$ over S such that $|\mathcal{A}_3| = |\mathcal{A}_1| \odot |\mathcal{A}_2|$.

In this construction, one proceeds in pretty much the same way as in the construction of automaton realizing the sum of $|\mathcal{A}_1|$ and $|\mathcal{A}_2|$. The only difference is in the construction of the initial polynomial $P_{0,3}$, where one defines $P_{0,3} = P_{0,1} \cdot \mathit{shift}(P_{0,2})$. It is quite easy to show that an automaton constructed in this way realizes the Hadamard product of $|\mathcal{A}_1|$ and $|\mathcal{A}_2|$. □

We shall now show that there exists a commutative semiring S such that the class of formal power series realized by alternating weighted automata over S is closed neither under reversal, nor under Cauchy product. More specifically, we shall show this for the semiring $\mathbb{B}[y]$ of polynomials in indeterminate y with coefficients in the Boolean semiring $\mathbb{B}$.

We have already explained the operation of Cauchy product in Chapter 1. Let us now briefly explain the reversal of formal power series. If $w = a_1 a_2 \ldots a_n$ is a word over an alphabet Σ, where $a_1, \ldots, a_n$ are symbols in Σ, we write w^R to denote the word $a_n a_{n-1} \ldots a_1$ over Σ. If r is a formal power series over a commutative semiring S and over an alphabet Σ, the *reversal* of r is a formal power series r^R in $S\langle\langle\Sigma^*\rangle\rangle$ such that

$$(r^R, w) = (r, w^R)$$

for every w in Σ^*.

Let $\Sigma = \{a, b, \#\}$ and let r_B be a formal power series in $\mathbb{B}[y]\langle\langle\Sigma^*\rangle\rangle$ such that $(r_B, a^i\#b^j) = (1 + y^i)^j$ for every pair of nonnegative integers i and j and $(r_B, w) = 0$ for every other word w in Σ^*. For each i and j, we can equivalently write $(r_B, a^i\#b^j) = \sum_{k=0}^{j} y^{ki}$. In Figure 3.2, we depict a two-mode alternating weighted automaton over $\mathbb{B}[y]$ that realizes the reversal of r_B. Moreover, the series r_B can be obtained as a Cauchy product of two formal power series r_1 and r_2 that can be realized by alternating weighted automata over $\mathbb{B}[y]$. Let us describe one possible choice of r_1 and r_2. For every pair of nonnegative integers i, j, let $(r_1, a^i\#b^j) = (y^i)^j$ and for every other word w in Σ^*, let $(r_1, w) = 0$. For every word w in $\{b\}^*$, let $(r_2, w) = 1$ and for every other word w in Σ^*, let $(r_2, w) = 0$. It is easy to see that $r = r_1 r_2$. In Figure 3.3, we depict a two-mode alternating weighted automaton over $\mathbb{B}[y]$ that realizes r_1. Trivially, r_2 is realized by an alternating weighted automaton over $\mathbb{B}[y]$ as well.

We see that the series r_B can be obtained as a reversal of a formal power series that is realized by an alternating weighted automaton over $\mathbb{B}[y]$. It can also be obtained as a Cauchy product of two formal power series that are realized by an alternating weighted automaton over $\mathbb{B}[y]$. We shall now prove that the series r_B is not realized by an alternating weighted automaton over $\mathbb{B}[y]$.

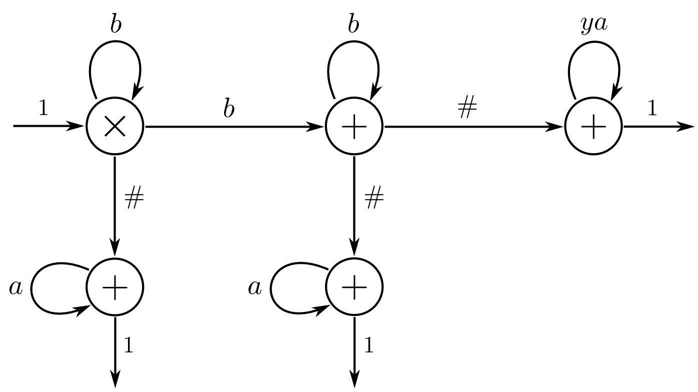

Figure 3.2: A two-mode alternating weighted automaton realizing the reversal of r_B.

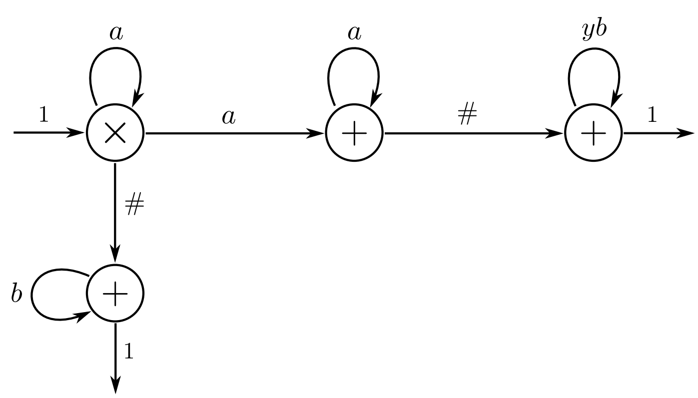

Figure 3.3: A two-mode alternating weighted automaton realizing series r_1 such that $(r_1, a^i\#b^j) = y^{ij}$ for every pair of nonnegative integers i and j and $(r_1, w) = 0$ for every other word w in $\{a, b, \#\}^*$.

As a result, the class of formal power series realized by alternating weighted automata over $\mathbb{B}[y]$ is closed neither under reversal nor under Cauchy product.

Before we prove this statement, some preparations need to be made. Let S be a commutative semiring and X an arbitrary set. We shall define a commutative semiring on the set S^X of all functions from X to S as follows: the sum of functions f_1, f_2 in S^X is a function $f_1 + f_2$ in S^X such that $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for every x in X. Similarly, the product of functions f_1, f_2 in S^X is a function $f_1 f_2$ in S^X such that $(f_1 f_2)(x) = f_1(x) f_2(x)$ for every x in X. The zero of the semiring S^X is a constant function h_0 such that $h_0(x) = 0_S$ for every x in X and the unity of the semiring S^X is a constant function h_1 such that $h_1(x) = 1_S$ for every x in X. In what follows, we shall work with the semiring $(\mathbb{B}[y])^{\mathbb{N}}$ with operations of sum and product as we have just described them.

Let S be a commutative semiring, X be an arbitrary set, and F be a set of functions from X to S. We say that functions in F *have the same support* if the following is true for every f_1, f_2 in F and every x in X: $f_1(x) = 0_S$ iff $f_2(x) = 0_S$.

For the purposes of the proof that follows, we shall also introduce the following terminology. Let S be a commutative semiring and P be a polynomial in $S[x_1, \ldots, x_n]$. The *number of terms* in a polynomial P, denoted by $\#terms(P)$, is the number of different monomials that occur in P. To be more precise, the number of terms in P is the smallest number k such that P can be obtained as a sum of k monomials. If m is a monomial in $S[x_1, \ldots, x_n]$, we shall write J_m to denote the set of indices i in $\{1, \ldots, n\}$ such that x_i occurs in m.

Lemma 3.2.3. *The formal power series r_B is not realized by an alternating weighted automaton over $\mathbb{B}[y]$.*

Proof. Suppose for the purpose of contradiction that r_B is realized by an alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ over $\mathbb{B}[y]$. Let $n = |Q|$. For $i = 1, \ldots, n$, let f_i be a map from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $f_i(k) = (|\mathcal{A}|_i, b^k)$ for every nonnegative integer k. For every nonnegative integer i, let g_i be a map from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $g_i(k) = (1 + y^i)^k$ for every nonnegative integer k.

For every nonnegative integer i, let $P_i = P_{\mathcal{A}}[a^i\#]$. We have

$$\begin{aligned} g_i(k) &= (|\mathcal{A}|, a^i\#b^k) = \\ &= P_{\mathcal{A}}[a^i\#]\big((|\mathcal{A}|_1, b^k), (|\mathcal{A}|_2, b^k), \ldots, (|\mathcal{A}|_n, b^k)\big) = \\ &= P_i(f_1(k), f_2(k), \ldots, f_n(k)) \end{aligned} \tag{3.16}$$

for every nonnegative integer k. Let us now interpret each coefficient c in P_i as a constant function h_c from $\mathbb{N}$ to $\mathbb{B}[y]$ with $h_c(k) = c$ for every nonnegative integer k. If we do this, P_i can be viewed as a polynomial in $(\mathbb{B}[y])^{\mathbb{N}}[x_1, \ldots, x_n]$. With this in mind, the equation (3.16) says that $P_i(f_1, \ldots, f_n) = g_i$ for every nonnegative integer i.

Let i be a nonnegative integer. The polynomial P_i can be written as a sum of some monomials with nonzero coefficients. For each such monomial m, we would like the functions in $\{f_j\}_{j \in J_m}$ to have the same support. Of course, this might not be the case. To make this true, we shall modify the polynomials $P_1, P_2, P_3, \ldots$ to polynomials $P'_1, P'_2, P'_3, \ldots$ in $(\mathbb{B}[y])^{\mathbb{N}}[x_1, \ldots, x_{n'}]$ (polynomials in n' indeterminates, where n' is some nonnegative integer) and replace the set of functions $F := \{f_1, \ldots, f_n\}$ with some other set of functions $F' := \{f'_1, \ldots, f'_{n'}\}$ in $(\mathbb{B}[y])^{\mathbb{N}}$. This modification needs to be done in such a way that we still have $P'_i(f'_1, \ldots, f'_{n'}) = g_i$ for every nonnegative integer i.

Let us first construct the set of functions F'. For $i = 1, \ldots, n$, let χ_i be a function from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $\chi_i(k) = 0$ if $f_i(k) = 0$ and $\chi_i(k) = 1$ otherwise. The set of functions F' consists of all functions $f \prod_{j \in J} \chi_j$, where f is in F and J is a subset of $\{1, \ldots, n\}$. Let n' be the size of F'. Let the functions in F' be somehow linearly ordered and let the i-th function of F' with respect to this ordering be denoted by f'_i.

Let i be a nonnegative integer. We shall now show how the polynomial P'_i can be constructed. We can write $P_i = \sum_{m \in M} c_m m$, where M is a finite set of monomials with coefficient 1 and c_m is a constant function in $(\mathbb{B}[y])^{\mathbb{N}}$ for each m in M. We have $g_i = \sum_{m \in M} c_m m(f_1, \ldots, f_n)$. Let m

be a monomial in M and let us look at the function $m(f_1, \ldots, f_n)$. By its definition, this function can be obtained as a product of some functions in F. For every j in J_m, let us replace each occurrence of f_j in this product with $f_j \prod_{k \in J_m} \chi_k$. We obtain a product of functions in F' that evaluates to $m(f_1, \ldots, f_n)$ and all functions in this product have the same support. It is now clear that we can find a monomial m' with coefficient 1 such that $m'(f'_1, \ldots, f'_{n'}) = m(f_1, \ldots, f_n)$ and functions in $\{f'_j\}_{j \in J_{m'}}$ have the same support. We can do this for every monomial m in M and so we conclude that we can construct a polynomial P'_i such that $P'_i(f'_1, \ldots, f'_{n'}) = P_i(f_1, \ldots, f_n) = g_i$ and functions in $\{f'_j\}_{j \in J_{m'}}$ have the same support for every monomial m' that occurs in P'_i.

Let J^∞ be the set of indices j in $\{1, \ldots, n'\}$ such that x_j occurs in P'_i for infinitely many indices i. We shall show that for every j in J^∞ and every nonnegative integer k, the number of terms in the polynomial $f'_j(k)$ in $\mathbb{B}[y]$ is at most 1 (i.e., either $f'_j(k) = y^i$ for some positive integer i, or $f'_j(k) = 1$, or $f'_j(k) = 0$). In order to obtain a contradiction, let us suppose that $f'_j(k) = y^{l_1} + y^{l_2} + B_1$ for some j in J^∞, nonnegative integers k, l_1 and l_2 satisfying $l_1 < l_2$, and B_1 in $\mathbb{B}[y]$. Let i be a nonnegative integer such that $i > l_2 - l_1$ and P'_i contains x_j (such i exists, since x_j occurs in infinitely many polynomials in $\{P'_1, P'_2, P'_3, \ldots\}$). Let m be a monomial in P'_i that contains x_j. We can write $m = x_j m'$, where m' is a nonzero monomial. We thus have

$$\begin{aligned} m[f'_1, \ldots, f'_{n'}](k) = f'_j(k) m'[f'_1, \ldots, f'_{n'}](k) = \\ = (y^{l_1} + y^{l_2} + B_1) m'[f'_1, \ldots, f'_{n'}](k). \end{aligned}$$

Since $f'_j(k)$ is nonzero and functions in $\{f'_j\}_{j \in J_m}$ have the same support, the value of $m'[f'_1, \ldots, f'_{n'}]$ at k is nonzero as well. We can write $m'[f'_1, \ldots, f'_{n'}](k) = y^{l_3} + B_2$, where l_3 is a nonnegative integer and B_2 is in $\mathbb{B}[y]$.[2] We have

$$\begin{aligned} m[f'_1, \ldots, f'_{n'}](k) &= (y^{l_1} + y^{l_2} + B_1) m'[f'_1, \ldots, f'_{n'}](k) = \\ &= (y^{l_1} + y^{l_2} + B_1)(y^{l_3} + B_2) = \\ &= y^{l_1 + l_3} + y^{l_2 + l_3} + B_3 \end{aligned}$$

for some B_3 in $\mathbb{B}[y]$. We conclude that $P'_i[f'_1, \ldots, f'_{n'}](k) = y^{l_1+l_3} + y^{l_2+l_3} + B_4$ for some B_4 in $\mathbb{B}[y]$ and

$$\begin{aligned} \sum_{j'=0}^{k} (y^i)^{j'} &= g_i(k) = \\ &= P'_i[f'_1, \ldots, f'_{n'}](k) = \\ &= y^{l_1+l_3} + y^{l_2+l_3} + B_4. \end{aligned}$$

This is clearly a contradiction, since $i > l_2 - l_1$.

We have thus proved that if j is in J^∞ and k is a nonnegative integer, then the number of terms in $f'_j(k)$ is at most 1. Let i be a nonnegative integer such that every indeterminate that occurs in P'_i is in $\{x_j\}_{j \in J^\infty}$. We can write $P'_i = \sum_{m \in M} c_m m$, where M is some finite set of monomials with coefficient 1 and c_m is a constant function in $(\mathbb{B}[y])^{\mathbb{N}}$. Let k be a nonnegative integer. We have $g_i(k) = \sum_{m \in M} c_m(k) m[f'_1, \ldots, f'_{n'}](k)$. Since the number of terms in $f'_j(k)$ is at most 1 for every j in J_m, the number of terms in $m[f'_1, \ldots, f'_{n'}](k)$ is also at most 1 for every monomial m in M. This implies that the number of terms in $c_m(k) m[f'_1, \ldots, f'_{n'}](k)$ is at most the number of terms in $c_m(k) = c_m(0)$ for every monomial m in M. We conclude that the number of terms in $g_i(k) = \sum_{m \in M} c_m(k) m[f'_1, \ldots, f'_{n'}](k)$ is at most $\sum_{m \in M} \#terms(c_m(0))$ for every nonnegative integer k. This is clearly a contradiction. □

We have thus proved the following two theorems.

Theorem 3.2.4. *There exists a commutative semiring S such that the class of formal power series realized by alternating weighted automata over S is not closed under reversal.*

[2] Note that $m'[f'_1, \ldots, f'_{n'}](k)$ might be $1 + B_2$ for some B_2 in $\mathbb{B}[y]$.

Proof. In Figure 3.2, we depicted a two-mode alternating weighted automaton over $\mathbb{B}[y]$ that realizes a formal power series r such that $r^R = r_B$. By Lemma 3.2.3, series r_B is not realized by an alternating weighted automaton over $\mathbb{B}[y]$. This means that the class of formal power series realized by alternating weighted automata over $\mathbb{B}[y]$ is not closed under reversal. □

Theorem 3.2.5. *There exists a commutative semiring S such that the class of formal power series realized by alternating weighted automata over S is not closed under Cauchy product.*

Proof. In Figure 3.3, we depicted a two-mode alternating weighted automaton over $\mathbb{B}[y]$ that realizes a formal power series r_1 over $\mathbb{B}[y]$ and $\Sigma = \{a, b, \#\}$ such that $(r_1, a^i \# b^j) = (y^i)^j$ for every pair of nonnegative integers i, j and $(r_1, w) = 0$ for every other w in Σ^*. A formal power series r_2 over $\mathbb{B}[y]$ and Σ such that $(r_2, w) = 1$ for every w in $\{b\}^*$ and $(r_2, w) = 0$ for every other w in Σ^* is trivially realized by an alternating weighted automaton over $\mathbb{B}[y]$ as well. The reader can easily check that $r_1 r_2 = r_B$. By Lemma 3.2.3, series r_B is not realized by an alternating weighted automaton over $\mathbb{B}[y]$. This means that the class of formal power series realized by alternating weighted automata over $\mathbb{B}[y]$ is not closed under Cauchy product. □

Conclusion

We have defined and initiated the study of alternating weighted automata over an arbitrary commutative semiring S. These form a new extension of alternating finite automata [2], in which transitions carry weights given by elements of S. Moreover, disjunctions are replaced by sums and conjunction are replaced by products of the semiring S. The behaviour of an alternating weighted automaton is thus a formal power series. We have seen that both alternating finite automata (without weights) [2] and (nonalternating) weighted automata [4] can be viewed as a special case of alternating weighted automata.

Thesis partially builds on the line of research initiated by Chatterjee, Doyen and Henzinger [3] and later continued by Almagor and Kupferman [1]. These two studies both focused on alternation in weighted automata over the tropical semiring and were motivated by certain problems in formal verification of reactive systems. Our goal has been to study alternating weighted automata over *general* commutative semirings and from a theoretical point of view.

We have given two alternative definitions of alternating weighted automata. In the first definition, each state was equipped with a set of polynomials so that it could combine additive and multiplicative operation. In the second definition, each state was either a sum state or a product state: a sum state could only perform addition and a product state could only perform multiplication. To be more precise, these two definitions introduced two different models: the first definition introduced automata simply called alternating weighted automata and the second definition introduced automata called two-mode alternating weighted automata. Nevertheless, we have proved that these two models are equivalent.

We have introduced the notion of a run tree, which allowed us to give an alternative characterization of the behaviour of two-mode alternating automata. Two-mode alternating weighted automata are allowed to contain ε-labelled transitions. Although these transitions might be quite handy, they are not strictly necessary: we have presented a construction that can be used to eliminate ε-labelled transitions in a two-mode alternating weighted automaton.

We have also introduced systems of H-polynomial equations, which provide us with a different characterization of formal power series realized by alternating weighted automata: a formal power series r is realized by an alternating weighted automaton iff it is the solution to some system of H-polynomial equations. The characterization of formal power series realized by alternating weighted automata in terms of systems of H-polynomial equations goes in the same lines as the well-known characterization of rational series in terms of linear systems [5].

We have studied the relationship between the expressive power of weighted automata on the one hand and the expressive power of alternating weighted automata on the other hand. In the most important result of this thesis, we prove that alternating weighted automata over a commutative semiring S have the same expressive power as (non-alternating) weighted automata over S iff the semiring S is such that every finitely generated subsemiring of S is finite. Otherwise, alternating weighted automata over S are strictly more powerful.

Finally, we have also examined some closure properties of the classes of formal power series realized by alternating weighted automata. We have proved that for every commutative semiring S, the class of formal power series realized by alternating weighted automata over S is closed under sum and Hadamard product. We have also shown that there exists a commutative semiring S such that the class of formal power series realized by alternating weighted automata over S is closed neither under reversal nor under Cauchy product.

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