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Infinite series: Convergence tests

(Bachelor thesis)

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Study programme: 9.2.1. Informatics
Supervisor: doc. RNDr. Zbyněk Kubáček, CSc. Bratislava, 2009

I hereby proclaim, that I worked out this bachelor thesis alone and using only quoted sources.

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Author would like to thank his supervisor doc. RNDr. Zbyněk Kubáček, CSc. for his kind help and guidance, Bc. Peter Lenčéš for extensive TEXnical help, Bc. Peter Beňák for many advices and Jozef Vojtek for language support.

Thou, nature, art my goddess; to thy laws my services are bound...

W. Shakespear, *King Lear*

motto of J. C. F. Gauss

Abstract

The theory of infinite series, an old and well examined part of calculus, gives us a powerfull tool for solving a wide range of problems such as evaluating important mathematical constants (e, π, ...), values of trigonometric functions etc.

This thesis introduces the idea of infinite series with some basic theorems necessary in the following chapters. Then building on this, it derives some convergence tests, namely Raabe's test, Gauss' test, Bertrand's test and Kummer's test.

It also proves that there is no universal comparison test for all series.

Keywords: Infinite series, convergence, divergence, Kummer, Gauss, Bertrand, Raabe.

Abstrakt

Teória nekonečných radov, stará a dobre preskúmaná oblasť matematickej analýzy, nám dáva silný nástroj na riešenie širokého spektra problémov, ako napríklad vyčíslenie známych matematických konštánt (e, π, ...), hodnôt trigonometrických funkcii atď.

Táto práca uvádza základné definície a teorémy z tejto oblasti, ktoré budú dôležité v ďalších kapitolách. Potom, stavajúc na týchto základoch, odvádza vybrané kritéria vyšetrujúce konvergenciu nekonečných radov, menovite Raabeho test, Gaussov test, Bertrandov test a Kummerov test.

Tiež dokážeme, že neexistuje univerzálne porovnávacie kritérium, ktoré by vedelo rohodnúť konvergenciu/divergenciu všetkých radov.

Kľúčové slová: Nekonečné rady, konvergencia, divergencia, Kummer, Gauss, Bertrand, Raabe.

Contents

Chapter 1

Prologue

1.1 The historical background of infinite series

The first mention of infinite series dates back to antiquity. In those times, to understand that a sum with infinite number of summands could have a finite result was an important philosophical challenge. Later, the theory of infinite series was thoroughly developed and used to work out many significant problems that eluded solutions with any other approach.

Greek Archimedes (c. 287 BC – c. 212 BC) was first known mathematician who applied infinite series to calculate the area under the arc of parabola using the method of exhaustion[1].

Several centuries later, Madhava (c.1350 – c.1425) from India came up with the idea of expanding functions into infinite series. He laid down the precursors of modern conception of power series, Taylor series, Maclaurin series, rational approximations of infinite series and infinite continued fractions.

Next important step was taken by Scottish mathematician James Gregory (1638 – 1675). Gregory understood the differential and integral, before it was formulated by Newton and Leibniz, on such a level that he was able to find the infinite series of arctangent by using his own methods. Even though this is the main result attributed to him, he discovered infinite series

[1]The method of exhaustion is a method of finding the area of a shape by inscribing inside it a sequence of polygons whose areas converge to the area of the containing shape. If the sequence is correctly constructed, the difference in area between the nth polygon and the containing shape will become arbitrarily small as n becomes large.[2]

[2]adopted from http://en.wikipedia.org/wiki/Method_of_exhaustion on 2nd May, 2009

for tangent, secant, arcsecant and some others as well.

After English Sir Isaac Newton ($1642 - 1727$) and German Gottfried Wilhelm Leibniz ($1646 - 1716$) independently developed and published formal methods of calculus, they achieved many discoveries within the theory of infinite series. But even such a genius as Leibniz was unable to find the sum of inverse squares

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$$

This problem was eventually resolved by Swiss Leonhard Euler ($1707 - 1783$). Euler solved this problem using infinite series and developed new ways to manipulate them.

With the discovery of infinite series there also came a question which of them actually yield a finite result. Obviously, the result of series $\sum_{k=1}^{\infty} k$ cannot be finite so how we can decide the more obscure ones? First mathematician to study convergence was Madhava in 14th century. He developed a test[3], which was further developed by his followers in the Kerala school.

In Europe the development of convergence tests was started by German Johann Carl Friedrich Gauss ($1777 - 1855$), but the terms of convergence and divergence had been introduced long before by J. Gregory. French Augustin Louis Cauchy ($1789 - 1857$) proved that a product of two convergent series does not have to be convergent, and with him begins the discovery of effective criterions, although his methods led to a special (applicable for a certain range of series) rather than a general criterions. And the same applies for Swiss Joseph Ludwig Raabe ($1801 - 1859$), British Augustus De Morgan ($1806 - 1871$), French Joseph Louis Francois Bertrand ($1822 - 1900$) and others.

Development of general criterions began with German Ernst Eduard Kummer ($1810 - 1893$) and later was carried on by German Karl Theodor Wilhelm Weierstrass(1815 - 1897), German Ferdinand Eisenstein ($1823 - 1852$) and many others.[5]

[3]Early form of an integral test of convergence; in Europe it was later developed by Maclaurin and Cauchy and is sometimes known as the Maclaurin - Cauchy test. [4]

[4]adopted from http://en.wikipedia.org/wiki/Integral_test_for_convergence on 2nd May, 2009

[5]text sources (on 2nd May, 2009):
http://en.wikipedia.org/wiki/Infinite_series
http://www.math.wpi.edu/IQP/BVCalcHist/calc3.html

1.2 Elementary definitions and theorems

In this section we introduce the basic definitions such as what exactly the infinite series are, what the convergence and divergence terms mean etc. Also we will get acquainted with a few important theorems which we will refer to in upcoming chapters. (All definitions were adopted from [1].)

Definition 1.2.1. Let $\{a_n\}_{n=k}^{\infty}$ ($k \in \mathbb{N} \cup \{0\}$) be a sequence. Then we call the symbol $\sum_{n=k}^{\infty} a_n$ (or $a_k + a_{k+1} + a_{k+2} + ... + a_n + ...$) a *(infinite) series*. Moreover, as we consider the symbols $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=m}^{\infty} a_{n+k-m}$ indentical, because we can write all series $\sum_{n=k}^{\infty} a_n$ in the form $\sum_{n=1}^{\infty} a_{n+k-1}$, all definitions and theorems will be formulated for the series in the form $\sum_{n=1}^{\infty} b_n$.

Definition 1.2.2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The number a_k, resp.

$$S_k = a_1 + a_2 + ... + a_k, \quad k \in \mathbb{N}$$

is called *k-th term*, resp. *k-th partial sum of series* $\sum_{n=1}^{\infty} a_n$.

Definition 1.2.3. We say that the series $\sum_{n=1}^{\infty} a_n$ *converges (is convergent)* if there exists a finite $\lim_{n\to\infty} S_n$. We call the number $\lim_{n\to\infty} S_n$ *the sum of series* $\sum_{n=1}^{\infty} a_n$ and we denote it with the same symbol $\sum_{n=1}^{\infty} a_n$ (or $a_1 + a_2 + ... + a_n + ...$).
If the $\lim_{n\to\infty} S_n$ is not finite then we say that *the series* $\sum_{n=1}^{\infty} a_n$ *diverges (is divergent)* and we can distinguish three cases:
1) if $\lim_{n\to\infty} S_n = +\infty$ we say that *the series* $\sum_{n=1}^{\infty} a_n$ *diverges to* $+\infty$;
2) if $\lim_{n\to\infty} S_n = -\infty$ we say that *the series* $\sum_{n=1}^{\infty} a_n$ *diverges to* $-\infty$;
3) if $\lim_{n\to\infty} S_n$ does not exist we say that *the series* $\sum_{n=1}^{\infty} a_n$ *oscillate.*

Definition 1.2.4. $\sum_{n=k+1}^{\infty} a_n$ is called *k-th remainder of series* $\sum_{n=1}^{\infty} a_n$.

Having become familiar with the most important definitions we can proceed to the formulation of some theorems. (All theorems were adopted from [2].)

Theorem 1.2.5. *The series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the following is true:*

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \in \mathbb{N} \quad n > N, \quad \forall p \in \mathbb{N}$$

$$|a_n + a_{n+1} + ... + a_{n+p}| < \epsilon.$$

Remark 1.2.6. This theorem is known as Cauchy-Bolzano's theorem.

Corollary 1.2.7. *(Necessary requirement for convergence)*

If the series $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Corollary 1.2.8. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges then*

$$\forall \epsilon > 0, \quad \exists k \in \mathbb{N} \quad : \quad \left| \sum_{n=k+1}^{\infty} a_n \right| < \epsilon.$$

Theorem 1.2.9. *Let at most finite count of numbers* n *fail* $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, *where* $a_n > 0$, $b_n > 0$, $n = 0, 1, 2, ...$ *Then the convergence of series* $\sum_{n=1}^{\infty} b_n$ *implies the convergence of series* $\sum_{n=1}^{\infty} a_n$ *and the divergence of series* $\sum_{n=1}^{\infty} a_n$ *implies the divergence of series* $\sum_{n=1}^{\infty} b_n$ *.*

Remark 1.2.10. Known as the second comparison test.

Theorem 1.2.11. *Let* $y = f(x)$ *be a continuous, non-negative, non-increasing function defined on the interval* $[1, \infty)$*. Let* $a_n = f(n)$ *for n=1, 2, ... and* $F(x)$ *be a primitive function to the function* $f(x)$ *on the interval* $[1, a]$*, where* $a > 1$ *is an arbitrary real number. Then:*
1) if $\lim_{x\to\infty} F(x)$ *is finite then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
2) if $\lim_{x\to\infty} F(x) = +\infty$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark 1.2.12. Known as the integral test of convergence.

Chapter 2

Joseph Ludwig Raabe

Born on May 15th, 1801, in Brody, Galicia, J. L. Raabe was a Swiss mathematician. He began to study mathematics in 1820 at the Polytechnicum in Vienna, Austria. In autumn 1831, he moved to Zurich, where he became a professor of mathematics in 1833. In 1855, he became a professor at the newly founded Swiss Polytechnicum.
Joseph Ludwig Raabe died on January 22nd, 1859, in Zurich, Switzerland.

His best known success is Raabe's test of convergence.[1]

[1] biography source (from 2nd May, 2009):
http://en.wikipedia.org/wiki/Joseph_Ludwig_Raabe

2.1 Raabe's ratio test

Let's consider the following sequence

$$a_n = \frac{1}{n^p} \quad p > 0 \quad n = 1, 2, ...$$

The first question we may ask is wether the series $\sum_{n=1}^{\infty} a_n$ (sometimes called Riemann's series) converges. At first glance we see that the necessary requirement for convergence $\lim_{n\to\infty} a_n = 0$ holds, so we need something more sophisticated.

Let's put this sequence into the integral criterion and see what happens. First, we define $f(x)$ as a function

$$f(x) = \frac{1}{x^p}$$

on the interval $[1, \infty)$, which satisfies $f(n) = a_n$ $(n = 1,\ 2,\ 3, ...)$. We check that the function $f(x)$ meets all the preconditions required for this test and we continue by finding a primitive function $F(x)$ to the function $f(x)$. After some easy calculations, we have the result:

$$F(x) = \begin{cases} \ln x & \text{if } \ p = 1 \\ \frac{1}{(1-p)x^{(p-1)}} & \text{if } \ p \neq 1 \end{cases}$$

Now we check the $\lim_{x\to\infty} F(x)$

$$\lim_{x\to\infty} F(x) = \begin{cases} K < \infty & \text{if } \ p > 1 \\ \infty & \text{if } \ p \in (0, 1] \end{cases}$$

and according to the integral test (theorem 1.2.11) we can conclude that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } \ p > 1 \\ \text{diverges} & \text{if } \ p \in (0, 1] \end{cases} \tag{2.1}$$

This was easy, but let's go on and see what we can learn about $\frac{a_n}{a_{n+1}}$. This might tell us something about how fast a_n approaches zero as n tends to infinity. Equation follows:

$$\frac{a_n}{a_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + O\left(\frac{1}{n^2}\right) \tag{2.2}$$

We used here

$$\left(1 + \frac{1}{x}\right)^p = 1 + \frac{p}{x} + O\left(\frac{1}{x^2}\right)$$

as an asymptotic approximation valid for $x \to \infty$. After some adjustments we get

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + O\left(\frac{1}{n}\right)$$

where the expresion $O(\frac{1}{n})$ becomes insignificant compared to p as n tends to infinity. Thus for sufficiently large n we can write

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \to p$$

Put together with (2.1), we can formulate the following theorem, which is known as Raabe's test:

Theorem 2.1.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms ($\forall k \in \mathbb{N}: \ a_k > 0$). If*

$$\exists p > 1, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ : \ n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq p \tag{2.3}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\exists N \in \mathbb{N}, \quad \forall n > N \ : \ n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1 \tag{2.4}$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Because we can write

$$p = q + \epsilon, \quad q > 1, \quad \epsilon > 0$$

and with (2.3) (for sufficiently large n):

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q + \epsilon \geq q + O\left(\frac{1}{n}\right)$$

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{q}{n} + O\left(\frac{1}{n^2}\right) = \left(1 + \frac{1}{n}\right)^q = \frac{\frac{1}{n^q}}{\frac{1}{(n+1)^q}}$$

And since $q > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ converges. According to the second comparison test (theorem 1.2.9) the series $\sum_{n=1}^{\infty} a_n$ converges as well.

On the other hand, adjusting (2.4) we get

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} = \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

Theorem 1.2.9 implies the divergence of series $\sum_{n=1}^{\infty} a_n$.

□

A more general version of this test is using the remainder $O(n^{-2})$, (which we formerly hid inside the constant p):

Theorem 2.1.2. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assume that there exists a real bounded sequence B_n such that for all n*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^2}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $p > 1$. The series diverges if and only if $p \leq 1$.

Proof. Let's assume that the series $\sum_{n=1}^{\infty} a_n$ converges and $p = 1$. Using Bertrand's test (theorem 4.1.1, see chapter four)

$$\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) = \frac{B_n \ln n}{n} \approx \frac{\ln n}{n}$$

With l'Hopital's rule we get $\frac{\ln n}{n} \to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges (according to Bertrand's test). A contradiction.

If $p < 1$ then (for sufficiently large n)

$$n \left(\frac{a_n}{a_{n+1}} - 1 \right) = p + \frac{B_n}{n} < 1$$

and according to the theorem 2.1.1 the series $\sum_{n=1}^{\infty} a_n$ diverges. A contradiction. Thus $p > 1$.

Now we assume that $p > 1$. For sufficiently large n

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^2} \leq 1 + \frac{q}{n}, \quad q > 1$$

We finish the proof with theorem 2.1.1. □

Divergence follows by analogy.

Theorem 2.1.3. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assume that there exists a real bounded sequence B_n such that for all n*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^2}$$

Then the series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$.

Proof. Let's consider a divergent series $\sum_{n=1}^{\infty} a_n$ and let $p > 1$. Then for sufficiently large n we get

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n} = q, \quad q > 1$$

which implies (according to the theorem 2.1.1), that the series $\sum_{n=1}^{\infty} a_n$ is actually convergent. A contradiction (hence $p \leq 1$).

Now for the part where $p \leq 1$. Let us assume that $p = 1$, for the divergence in case $p < 1$ is obvious. By using Bertrand's test we get

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = \ln n\left(n\left(1 + \frac{1}{n} + \frac{B_n}{n^2} - 1\right) - 1\right) = \frac{B_n \ln n}{n}$$

Because B_n is bounded

$$\frac{B_n \ln n}{n} \approx \frac{\ln n}{n}$$

With l'Hopital's rule we get $\frac{\ln n}{n} \to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges. □

An example. We want to determine the character of series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$.

$$\frac{a_n}{a_{n+1}} = \frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}} = \frac{2n+2}{2n+1}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{2n+2}{2n+1} - 1\right) = \frac{n}{2n+1} \leq 1$$

According to Raabe's test, the series diverges.

Chapter 3

Johann Carl Friedrich Gauss

Born on 30th April, 1777, in Braunschweig, in the Electorate of Brunswick-Luneburg (now part of Lower Saxony), Germany, J. C. F. Gauss was a German mathematician. He is known as the *princeps mathematicorum* which means the prince of mathematicians. And verily this title suits him up to the hilt. He learnt to read and count by the age of three. At elementary school when given a task to sum the integers from 1 to 100, he produced a correct answer within seconds.

His prodigious mind was quickly recognized by his teachers. Supported by his mother and by Duke of Brunswick - Wolfenbuttel, who gave him scholarship, Gauss entered the Brunswick Collegium Carolinum (1792 to 1795), and subsequently he moved to the University of Gottingen(1795 to 1798). While studying at university, he rediscovered several important theorems, including Bode's law, the binomial theorem, the law of quadratic reciprocity and the prime number theorem. His breakthrough came in 1796, when he showed that any regular polygon with a number of sides that equals a Fermat prime can be constructed by compass and straightedge. This was the most important advancement in this field since the time of Greek mathematicians.

In 1799, he proved fundamental theorem of algebra, that every non-constant single-variable polynomial over the complex numbers has at least one root. In 1801, he published a book *Disquisitiones Arithmeticae* (Arithmetical Investigations). In the same year, Gauss helped astronomer Giuseppe

Piazzi with observation of a small planet Ceres by predicting its position. This opened him the doors to astronomy. His brilliant work published as *Theory of Celestial Movement* remains a cornerstone of astronomical computation. In 1807, he was appointed Professor of Astronomy and Director of the astronomical observatory in Gottingen.

In 1818, while carring out a geodesic survey of the state of Hanover, he invented a heliotrope, an instrument for measuring positions by reflecting sunlight over great distances. He also discovered the possibility of non-Euclidean geometries, though he never published it because of fear of controversy. In 1828, he published *Disquisitiones generales circa superficies curva*, a work on differential geometry. His famous theorem *Theorema Egregium* (Remarkable Theorem) from this publication states (in a broad language), that the curvature of a surface can be determined by measuring angles and distances on the surface only.

In his late years, Gauss collaborated with physicist professor Wilhelm Weber investigating the theory of terrestrial magnetism. By the year 1840, he published three important papers on this subject. *Allgemeine Theorie des Erdmagnetismus* (1839) showed that there can be only two poles in the globe and specified the location of the magnetic South pole.

Together with Weber they discovered Kirchhoff's circuit laws in electricity and constructed electromagnetic telegraph, which was able to send a message over 5000 feet distance. But Gauss was far more interested in magnetic field of Earth. This lead to establishing a world-wide net of magnetic observation points, founding of *The Magnetischer Verein* (The Magnetic Club) and publishing the atlas of geomagnetism. He developed a method of measuring the horizontal intensity of the magnetic field which was in use for more than one hundred following years and worked out the mathematical theory for separating the inner (core and crust) and outer (magnetospheric) sources of Earth's magnetic field.

After the year 1837, his activity gradually decreased . In 1849, he presented his golden jubilee lecture, 50 years after his diploma.

Johann Carl Friedrich Gauss died on 23rd February, 1855, in Gottingen, Hannover (now part of Lower Saxony, Germany), in his sleep.[1]

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[1] biography sources (from 2nd May, 2009):
http://en.wikipedia.org/wiki/Gauss
http://www-groups.dcs.st-and.ac.uk/history/Printonly/Gauss.html

3.1 Gauss' test

Theorem 3.1.1. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with positive terms. Assuming that there exist a real number* p*, a real number* $r > 1$ *and a real bounded sequence* $\{B_n\}_{n=1}^{\infty}$ *such that for all* n

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^r}$$

Then the series $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $p > 1$.[2]

Almost immediately we see that this test is somehow an improved version of Raabe's test. The only difference is the number r, which in the former test was explicitly set to 2. So Gauss' test is more general, allowing us to decide convergence of more series. Not suprisingly the proof is very similar.

Proof. If we start with $p > 1$, then for sufficiently large n we have (for negative terms of $\{B_n\}$)

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n^{r-1}} \geq q, \quad q \in (1, p)$$

or (for positive terms of $\{B_n\}$)

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n^{r-1}} \geq p$$

Since in both cases

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q > 1$$

by using Raabe's test (theorem 2.1.1) we can conclude that the series $\sum_{n=1}^{\infty} a_n$ converges.

As for the other part of equivalence, we will assume that the series $\sum_{n=1}^{\infty} a_n$ converges and $p = 1$. By using Bertrand's test (theorem 4.1.1) we get

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = \ln n\left(n\left(1 + \frac{1}{n} + \frac{B_n}{n^r} - 1\right) - 1\right) =$$

$$= \frac{B_n \ln n}{n^{r-1}} \approx \frac{\ln n}{n^{r-1}}$$

[2]adopted from http://math.feld.cvut.cz/mt/txte/2/txe3ea2d.htm on 2nd May, 2009

With l'Hopital's rule we get $\frac{\ln n}{n^{r-1}} \to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges. A contradiction.

If the series $\sum_{n=1}^{\infty} a_n$ converges and $p < 1$ then (for sufficiently large n)

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n^{r-1}} < 1$$

and according to Raabe's test (theorem 2.1.1) the series $\sum_{n=1}^{\infty} a_n$ diverges. A contradiction. Thus $p > 1$. $\square$

Theorem 3.1.2. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assuming that there exist a real number p, a real number $r > 1$ and a real bounded sequence B_n such that for all n*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \frac{B_n}{n^r} \tag{3.1}$$

Then the series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$.

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series and $p > 1$. Then for sufficiently large n

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n^{r-1}} > q, \quad q > 1$$

and by Raabe's test (theorem 2.1.1) the series $\sum_{n=1}^{\infty} a_n$ converges. A contradiction (hence $p \leq 1$).

If $p = 1$ then by using Bertrand's test (theorem 4.1.1)

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = \frac{B_n \ln n}{n^{(r-1)}}$$

With l'Hopital's rule we get $\frac{\ln n}{n^{r-1}} \to 0$ (as n tends to infinity) and therefore the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $p < 1$ then for sufficiently large n

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = p + \frac{B_n}{n^{r-1}} < 1$$

We finish the proof with Raabe's test. $\square$

Now we show how to make a good use of mysterious looking sequence B_n (that is, how to reduce our thinking and use an algorithmic procedure instead). Look again at (3.1). The best p can be obtained by a limit

$$p = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) \tag{3.2}$$

Moreover, we set

$$A_n = \frac{a_n}{a_{n+1}} - 1 - \frac{p}{n} \tag{3.3}$$

Now we try to find a number r such that

$$r > 1, \quad B_n = A_n n^r < \infty \tag{3.4}$$

That is, $\{B_n\}$ is a bounded sequence. If we succeed, we can apply Gauss' test.[3]

An example. We want to determine the character of the series $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^a$, $a > 0$.

$$\frac{a_n}{a_{n+1}} = \left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}}\right)^a = \left(\frac{2n+2}{2n+1}\right)^a$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\left(\frac{2n+2}{2n+1}\right)^a - 1\right) = n\left(\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right)^a - 1\right)$$

With $(x \to \infty)$

$$\left(1 + \frac{1}{x}\right)^p = 1 + \frac{p}{x} + o\left(\frac{1}{x}\right)$$

we get the final result

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{1 + \frac{a}{n} + o\left(\frac{1}{n}\right)}{1 + \frac{a}{2n} + \left(\frac{1}{n}\right)} - 1\right) = \frac{a}{2} + o(1)$$

According to Raabe's test, the series converges when $a > 2$ and diverges when $a < 2$ but we get no information when $a = 2$. We try Gauss' test :

First, we try to find p (see (3.2))

$$p = \lim_{n\to\infty} n\left(\left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}}\right)^2 - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{2n+2}{2n+1}\right)^2 - 1\right) = 1$$

[3]adopted from http://math.feld.cvut.cz/mt/txte/2/txe3ea2d.htm on 2nd May, 2009

Second, we find A_n (see (3.3))

$$A_n = \frac{a_n}{a_{n+1}} - 1 - \frac{p}{n} = \left(\frac{\frac{(2n-1)!!}{(2n)!!}}{\frac{(2n+1)!!}{(2n+2)!!}} \right)^2 = \left(\frac{2n+2}{2n+1} \right)^2 - 1 - \frac{1}{n}$$

$$A_n = \frac{-n-1}{4n^3+4n^2+n}$$

Third, we try to find $r > 1$ such that $B_n = A_n n^r$ is bounded (see (3.4))

$$A_n = \frac{-n-1}{4n^3+4n^2+n} = \frac{1}{n^2} \left(\frac{-n^3-n^2}{4n^3+4n^2+n} \right)$$

If we set $r = 2$ then

$$B_n = A_n n^2 = \frac{-n^3-n^2}{4n^3+4n^2+n} < \infty$$

And because $p = 1$ the series $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2$ diverges.

Therefore, according to Gauss and Raabe's test, the series $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^a$ converges when $a > 2$ and diverges when $a \leq 2$.

Chapter 4

Joseph Louis Francois Bertrand

Born on 11th March, 1822, in Paris, J. L. F Bertrand was a French mathematician. He studied at Ecole Polytechnique and at the age of 16, he was awarded his first degree. In following year, he received his doctorate for a thesis on thermodynamics.

In 1845, Bertrand conjectured that there is at least one prime between n and $2n-2$ for every $n > 3$. The conjecture was later proved by Pafnuty Lvovich Chebyshev and it is now known as Bertrand's postulate. He made major contribution to the group theory and published works on differential geometry, on probability theory (Bertrand's paradox[1]) and on game theory (Bertrand paradox). He was also famous for writing textbooks mostly for pupils at secondary schools but later also for more advanced students.

In 1856, Bertrand was appointed a professor at Ecole Polytechnique and also a member of the Paris Academy of Sciences. In 1862, he was made a professor at College de France.

Joseph Louis Francois Bertrand died on 5th April, 1900, in Paris.[2]

[1] Bertrand's paradox concerns the probability that an arbitrary chord of a circle is longer than a side of an equilateral triangle inscribed in the circle.

[2] biography sources (from 2nd May, 2009):
http://en.wikipedia.org/wiki/Joseph_Louis_Francois_Bertrand
http://www-groups.dcs.st-and.ac.uk/history/Printonly/Bertrand.html

4.1 Bertrand's test

We begin with the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad p > 0 \tag{4.1}$$

First we want to know for which p the given series converges. To use integral criterion (theorem 1.2.11), we define $f(x)$ as a function

$$f(x) = \frac{1}{x(\ln x)^p}$$

on the interval $(1, \infty)$, which satisfies $f(n) = a_n$ $(n = 1, 2, 3, ...)$. Then the primitive function $F(x)$ is

$$F(x) = \int \frac{1}{x(\ln x)^p} dx = \frac{\ln x}{(\ln x)^p} - p \int \frac{\ln x}{x(\ln x)^{p+1}} dx$$

After some adjustments, we have the result

$$F(x) = \begin{cases} \ln \ln x & \text{if } \ p = 1 \\ \frac{1}{(1-p)(\ln x)^{p-1}} & \text{if } \ p \neq 1 \end{cases}$$

and from that we can conclude (owing to $\lim_{x\to\infty} F(x)$), that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \begin{cases} \text{converges} & \text{if } \ p > 1 \\ \text{diverges} & \text{if } \ p \in (0, 1] \end{cases} \tag{4.2}$$

Now we analyze $\frac{a_n}{a_{n+1}}$:

$$\frac{a_n}{a_{n+1}} = \frac{\frac{1}{n(\ln n)^p}}{\frac{1}{(n+1)(\ln (n+1))^p}} =$$

$$= \left(1 + \frac{1}{n}\right) \left(\frac{\ln (n+1)}{\ln n}\right)^p = \left(1 + \frac{1}{n}\right) \left(\frac{\ln n + \ln \left(1 + \frac{1}{n}\right)}{\ln n}\right)^p$$

Using little-o notation for an asymptotic approximation when x tends to infinity:

$$\ln \left(1 + \frac{1}{x}\right) = \frac{1}{x} + o\left(\frac{1}{x}\right)$$

$$\left(1 + \frac{1}{x}\right)^p = 1 + \frac{p}{x} + o\left(\frac{1}{x}\right)$$

We put these two formulas into $\frac{a_n}{a_{n+1}}$

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n}\right)\left(\frac{\ln n + \frac{1}{n} + o\left(\frac{1}{n}\right)}{\ln n}\right)^p =$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\ln n} + o\left(\frac{1}{n\ln n}\right)\right)^p =$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\ln n} + o\left(\frac{1}{n\ln n}\right)\right) =$$

$$= 1 + \frac{1}{n} + \frac{p}{n\ln n} + o\left(\frac{1}{n\ln n}\right) \tag{4.3}$$

and after some adjustments we get

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = p + o\,(1) \quad \rightarrow \quad p$$

Putting all this together, we can formulate a new test of convergence, which is known as Bertrand's test:

Theorem 4.1.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. If*

$$\exists p > 1, \quad \exists N \in \mathbb{N}, \quad \forall n > N \quad : \quad \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \geq p \tag{4.4}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\exists N \in \mathbb{N}, \quad \forall n > N \quad : \quad \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \leq 1 \tag{4.5}$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Truly, (4.4) implies the existence of two constants

$$q > 1, \quad \epsilon > 0 \ : \quad q + \epsilon = p$$

Therefore we can write (for sufficiently large n)

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \geq q + \epsilon \geq q + o(1)$$

with (4.3)

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{q}{n \ln n} + o\left(\frac{1}{n \ln n}\right) = \frac{\frac{1}{n(\ln n)^q}}{\frac{1}{(n+1)(\ln (n+1))^q}}$$

And because $q > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^q}$ is convergent (see (4.2)). According to the second comparison test (theorem 1.2.9) the series $\sum_{n=1}^{\infty} a_n$ converges as well.

Now to prove divergence part. Note that we cannot directly use equation (4.3) here (as we did in convergence part) because there can be hidden a negative function $f(n)$ in $o\left(\frac{1}{n \ln n}\right)$, which would make Betrand's test unusable:
From (4.3)

$$\frac{\frac{1}{n \ln n}}{\frac{1}{(n+1) \ln (n+1)}} = 1 + \frac{1}{n} + \frac{1}{n \ln n} + o\left(\frac{1}{n \ln n}\right) = 1 + \frac{1}{n} + \frac{1}{n \ln n} - f(n)$$

$$\frac{\frac{1}{n \ln n}}{\frac{1}{(n+1) \ln (n+1)}} \leq 1 + \frac{1}{n} + \frac{1}{n \ln n}$$

From (4.5):

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n \ln n}$$

Because the numbers $\frac{a_n}{a_{n+1}}$ can possibly be found between the numbers

$$\left(\frac{\frac{1}{n \ln n}}{\frac{1}{(n+1) \ln (n+1)}}\right) \quad \text{and} \quad \left(1 + \frac{1}{n} + \frac{1}{n \ln n}\right)$$

we do not have yet enough information to compare

$$\frac{a_n}{a_{n+1}} \quad \text{with} \quad \frac{\frac{1}{n \ln n}}{\frac{1}{(n+1) \ln (n+1)}}$$

What we need is something like equation (4.3) but without the $o\left(\frac{1}{n \ln n}\right)$ term.

$$\frac{\frac{1}{n \ln n}}{\frac{1}{(n+1) \ln (n+1)}} = \frac{(n+1) \ln (n+1)}{n \ln n} = \frac{(n+1)(\ln n + \ln\left(1 + \frac{1}{n}\right))}{n \ln n} =$$

$$= \frac{(n+1) \ln n + (n+1) \ln\left(1 + \frac{1}{n}\right)}{n \ln n} = 1 + \frac{1}{n} + \frac{(n+1) \ln\left(1 + \frac{1}{n}\right)}{n \ln n} =$$

$$= 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{(n+1)\ln\left(1+\frac{1}{n}\right)}{n \ln n} - \frac{1}{n \ln n} =$$

$$= 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{(n+1)\ln\left(1+\frac{1}{n}\right) - 1}{n \ln n} = 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{\epsilon(n)}{n \ln n}$$

Where

$$\epsilon(n) = (n+1)\ln\left(1 + \frac{1}{n}\right) - 1 \approx \frac{1}{2n} + o\left(\frac{1}{n}\right) \geq 0$$

Only now we have enough information to write

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n \ln n} \leq 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{\epsilon(n)}{n \ln n} = \frac{\frac{1}{n \ln n}}{\frac{1}{(n+1)\ln(n+1)}}$$

The second comparison test implies the divergence of the series $\sum_{n=1}^{\infty} a_n$ because the series $\sum \frac{1}{n \ln n}$ diverges (see (4.2)). $\square$

Remark 4.1.2. The reader should start to suspect that when we formulated Bertrand's test we commited a small imprecision. Luckily for us, this imprecision only weakened Bertrand's test (but formulation (4.5) is easy to use). As a result of this flaw, Bertrand's test cannot decide convergence of his own series (4.1), as it will be shown in Chapter 6.

Another version of Bertrand's test can be constructed by using the same Gauss' improvement as with Raabe's test:

Theorem 4.1.3. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assume that there exist a real number p, a real number $r > 1$ and a real bounded sequence B_n such that for all n*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{p}{n \ln n} + \frac{B_n}{n(\ln n)^r}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $p > 1$.

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and $p = 1$. According to the theorem 7.0.5 (see the Last chapter)

$$\ln\ln n\left(\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)-1\right)=\frac{B_n\ln\ln n}{(\ln n)^{r-1}}\approx\frac{\ln\ln n}{(\ln n)^{r-1}}$$

(using l'Hopital's rule) we get a contradiction with the convergence we assumed.

If $p < 1$ then for sufficiently large n

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)=p+\frac{B_n}{(\ln n)^{r-1}}<1$$

According to the theorem 4.1.1 we get a contradiction with the convergence we assumed. Thus $p > 1$.

If $p > 1$ then

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{q}{n\ln n},\quad q>1$$

with theorem 4.1.1 we can conclude the convergence of series $\sum_{n=1}^{\infty} a_n$. $\square$

Divergence follows by analogy.

Theorem 4.1.4. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Assume that there exist a real number p, a real number $r > 1$ and a real bounded sequence B_n such that*

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{p}{n\ln n}+\frac{B_n}{n(\ln n)^r}\tag{4.6}$$

Then the series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$.

Proof. Let $p = 1$. According to the theorem 7.0.5

$$\ln\ln n\left(\ln n\left(n\left(1+\frac{1}{n}+\frac{1}{n\ln n}+\frac{B_n}{n(\ln n)^r}-1\right)-1\right)-1\right)=$$

$$=\frac{B_n\ln\ln n}{(\ln n)^{(r-1)}}\quad\rightarrow\quad 0$$

(using l'Hopital's rule) we can conclude the divergence of series $\sum_{n=1}^{\infty} a_n$.

If $p < 1$ then for sufficiently large n

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)=p+\frac{B_n}{(\ln n)^{r-1}}<1$$

and the theorem 4.1.1 prooves the divergence of series $\sum_{n=1}^{\infty} a_n$.

Now let $\sum_{n=1}^{\infty} a_n$ be a divergent series. The theorem 4.1.1 with $p > 1$ contradicts the assumed divergence. Thus $p \leq 1$. $\square$

Remark 4.1.5. Again, we can use B_n and r to algorithmize the procedure. With

$$p = \lim_{n \to \infty} \ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right), \quad A_n = \frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} - \frac{p}{n \ln n}$$

we can try to find such $r > 1$, that $B_n = A_n n (\ln n)^r$ will be bounded. If we succeed, we can apply this test.

Chapter 5

On the universal criterion

In the chapters 2, 3 and 4 we dealt with various comparison tests (but for each one of them we can construct a series, that by using this test we get no information about the character of this series). That is, with tests that combine the second comparison test (theorem 1.2.9) and some series, whose convergence/divergence we know. Now we ask whether there can be made some fixed criterion, which will resolve the character of all series (with positive terms). Unfortunately, no such criterion can be constructed. The following two theorems will explain why. (Both theorems were adopted from [1].)

Theorem 5.0.6. *If $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms then there exists a monotonous sequence $\{B_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} B_n = \infty$ and series $\sum_{n=1}^{\infty} a_n B_n$ converges.*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. From the corollary 1.2.7 we have

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$

From the corollary 1.2.8 we have the numbers ξ_n, where $\{\xi_n\}_{n=1}^{\infty}$ is an increasing subsequence of natural numbers such that

$$\left(\sum_{k>\xi_n} a_k\right) \frac{1}{a_n} < a_n$$

Let us construct the numbers B_n this way:

$$B_n = \begin{cases} 0 & \text{if } \ n \in [1, \xi_1) \\ \frac{1}{a_k} & \text{if } \ n \in [\xi_k, \xi_{k+1}) \end{cases}$$

Now we only need to show that $\sum_{n=1}^{\infty} a_n B_n$ is convergent:

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{k=1}^{\xi_1 - 1} 0 a_k + \sum_{n=1}^{\infty} \left(\frac{1}{a_n} \sum_{k=\xi_n}^{\xi_{n+1}-1} a_k \right) \leq \sum_{n=1}^{\infty} a_n$$

□

Theorem 5.0.7. *If $\sum_{n=1}^{\infty} a_n$ is a divergent series with positive terms then there exists a monotonous sequence $\{B_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} B_n = 0$ and the series $\sum_{n=1}^{\infty} a_n B_n$ diverges.*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series. Here we consider two cases. First, if

$$\limsup_{n\to\infty} a_n \geq \epsilon > 0$$

then by leaving out all $a_n < \frac{\epsilon}{2}$ we do not change the character of series $\sum_{n=1}^{\infty} a_n$. We find all indices n such that $a_n \geq \frac{\epsilon}{2}$ and denote them with ξ_k. Thus $\{\xi_k\}_{k=1}^{\infty}$ is an increasing subsequence of natural numbers. By $\{B_n\}_{n=1}^{\infty}$ we denote such a sequence that

$$B_n = \begin{cases} 0 & \text{if } n \neq \xi_k \text{ for all } k \\ \frac{1}{k} & \text{if } n = \xi_k \text{ for some } k \end{cases}$$

Moreover $\lim_{n\to\infty} B_n = 0$ and

$$\sum_{n=1}^{\infty} a_n B_n \geq \sum_{n=1}^{\infty} \frac{\epsilon}{2n} = \infty$$

Second, if

$$\lim_{n\to\infty} a_n = 0$$

then from the divergence of $\sum_{n=1}^{\infty} a_n$ we have an such increasing subsequence $\{\xi_n\}_{n=1}^{\infty}$ of natural numbers that

$$\sum_{k=\xi_n}^{\xi_{n+1}-1} a_k > n, \quad \xi_1 = 1$$

If we let

$$\Xi_n = \sum_{k=\xi_n}^{\xi_{n+1}-1} a_k$$

then the series $\sum_{n=1}^{\infty} \frac{1}{n}\Xi_n$ diverges. We construct the numbers B_n this way

$$B_n = \frac{1}{k} \qquad \text{for } n \in [\xi_k, \xi_{k+1}),\ k = 1, 2, ...$$

Now we only need to show that the series $\sum_{n=1}^{\infty} a_n B_n$ is divergent:

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=\xi_n}^{\xi_{n+1}-1} a_k \right) = \sum_{n=1}^{\infty} \frac{1}{n}\Xi_n \geq \sum_{n=1}^{\infty} 1 = \infty$$

□

We showed that to any convergent (resp. divergent) series $\sum a_n$ with positive terms we can construct another positive series $\sum b_n$, that converges (resp. diverges) much more slowly. In other words, for any positive convergent (resp. divergent) series $\sum a_n$, there exists some positive series $\sum b_n$ that converges (resp. diverges) and the second comparison test (theorem 1.2.9) does not hold (or cannot be used). This implies that there can be no ultimate comparison test (that is, a test based on some fixed series) as all comparison tests are based on the theorem 1.2.9.

But what we can tell from this is that for any positive series $\sum a_n$ there can be constructed such a criterion (based on some known series $\sum b_n$), which will decide convergence/divergence of series $\sum a_n$, although finding such series $\sum b_n$ might be difficult. And that is how Kummer's test works.

Again in plain language, for all positive series there exists a comparison test which can decide its character. But there is no such fixed comparison test that can decide the character of all positive series.

Chapter 6

Ernst Eduard Kummer

Born on 29th January, 1810, in Sorau, Brandenburg (then part of Prussia, now Germany), E. E. Kummer was a German mathematician. He entered the University of Halle with the intention to study philosophy, but after he won a prize for his mathematical essay in 1831, he deciced for the mathematics. In the same year, he was awarded with a certificate enabling him to teach in schools and, for the strength of his essay, also a doctorate.

For ten years he was teaching mathematics and physics at the Gymnasium in Liegnitz, now Legnica in Poland. In 1836, he published a paper on hypergeometric series in Crelle's Journal and this work led Carl G. J. Jacobi together with Peter G. L. Dirichlet to correspond with him. In 1839, Kummer was elected to the Berlin Academy of Science.

In 1842, he was appointed a professor at the University of Breslau, now Wroclaw in Poland, and he began his reaserch in number theory. In 1855, Kummer became a professor of mathematics at the University of Berlin. With Karl T. W. Weierstrass and Leopold Kronecker, Berlin became one of the leading mathematical centres in the world.

Kummer helped to advance function theory and number theory. As for the function theory, he extended Gauss' work on hypergeometric functions, giving developments that are useful in the theory of differential equations. In number theory, he introduced the concept of "ideal" numbers. This work was fundamental to Fermat's last theorem and also to the development of the ring theory.

In 1857, he was awarded the Grand Prix by Paris Academy of Sciences for his work on Fermat's last theorem and soon after that, he was given a membership of the Paris Academy of Sciences. In 1863, he was elected a Fellow of the Royal Society of London. A year later, he published a work on Kummer surface, a surface that has 16 isolated conical double points and 16 singular tangent planes.

Ernst Eduard Kummer died on 14 May, 1893, in Berlin.[1]

[1]biography sources (from 2nd May, 2009):
http://en.wikipedia.org/wiki/Ernst_Kummer
http://www-groups.dcs.st-and.ac.uk/history/Printonly/Kummer.html
http://fermatslasttheorem.blogspot.com/2006/01/ernst-eduard-kummer.html

6.1 Kummer's test

Here comes probably the most powerful test for convergence, since it applies to all series with positive terms.[2] :

Theorem 6.1.1. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Then the series converges if and only if there exist a positive number A, positive numbers p_n and a number $N \in \mathbb{N}$ such that for all $n > N$*

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \geq A \tag{6.1}$$

The series diverges if and only if there exist positive numbers p_n such that $\sum \frac{1}{p_n} = \infty$ and a number $N \in \mathbb{N}$ such that for all $n > N$

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \leq 0 \tag{6.2}$$

Proof. First we prove the convergence. For the right-to-left implication, we adjust the equation (6.1)

$$p_n a_n - p_{n+1} a_{n+1} \geq A a_{n+1}$$

With $q_n = \frac{p_n}{A}$ we can write

$$q_n a_n - q_{n+1} a_{n+1} \geq a_{n+1} \tag{6.3}$$

Since we know the left side of upper inequality, we can construct a sequence $\{B_n\}_{n=1}^{\infty}$ such that

$$q_n a_n - q_{n+1} a_{n+1} = B_{n+1} a_{n+1}, \qquad \forall n \;\; B_n \geq 1$$

The sequence $\{q_n a_n\}_{n=1}^{\infty}$ is positive and decreasing (follows from (6.3)). Therefore it has a limit

$$0 \leq \lim_{n\to\infty} a_n q_n < q_1 a_1$$

Thus the series

$$\sum_{n=1}^{\infty} a_n B_n = \sum_{n=1}^{\infty} (q_n a_n - q_{n+1} a_{n+1}) = q_1 a_1 - \lim_{n\to\infty} q_n a_n > 0$$

[2]adopted from http://math.feld.cvut.cz/mt/txte/2/txe3ea2d.htm on 2nd May, 2009

converges. And because $B_n \geq 1$ for all n

$$a_n \leq a_n B_n \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} a_n B_n$$

In words, the series $\sum_{n=1}^{\infty} a_n B_n$ converges and creates an upper bound for the series $\sum_{n=1}^{\infty} a_n$, therefore the series $\sum_{n=1}^{\infty} a_n$ must converge as well. This is nothing more then just a first comparison criteria. We showed that finding numbers p_n is equally hard as finding a convergent series $\sum b_n$ which creates an upper bound for the series $\sum a_n$ in the first comparison test.

For left-to-right implication, let $p_1 a_1$ be a positive number. According to the theorem 5.0.6, we assume the existence of positive monotonous sequence $\{B_n\}_{n=1}^{\infty}$

$$\lim_{n \to \infty} B_n = \infty \tag{6.4}$$

$$\sum_{n=1}^{\infty} a_n B_n = p_1 a_1 + a_1 B_1$$

Now we shift the index n

$$\sum_{n=1}^{\infty} a_{n+1} B_{n+1} = p_1 a_1$$

We define the sequence $\{p_n a_n\}_{n=1}^{\infty}$ this way

$$p_{n+1} a_{n+1} = p_n a_n - a_{n+1} B_{n+1}$$

where

$$\lim_{n \to \infty} p_n a_n = p_1 a_1 - \lim_{n \to \infty} \sum_{k=1}^{n} a_{k+1} B_{k+1} = 0$$

So, using (6.4) we have (for sufficiently large n)

$$p_n a_n - p_{n+1} a_{n+1} = a_{n+1} B_{n+1} \geq A a_{n+1} \quad \text{where} \quad A > 0$$

And finally

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \geq A$$

hence the numbers p_n are found. □

Remark 6.1.2. The reader surely perceives that, in fact, the requirement (6.4) is not necessary as any positive and monotonous sequence $\{B_n\}$ with $\lim_{n\to\infty} B_n = A \geq 1$ is totally sufficient (where A is an arbitrary constant).

For example, let $\sum a_n$ be a convergent series. If we let $B_n = 1$ for all n and we construct numbers p_n using the terms from $\sum a_n$

$$p_{n+1} = p_n \frac{a_n}{a_{n+1}} - 1$$

Kummer's test will confirm the convergence of series $\sum a_n$ (as would first comparison test).

Now the divergence part.

Proof. For the right-to-left implication, we have $\sum \frac{1}{p_n} = \infty$ and from (6.2) we get

$$\frac{a_n}{a_{n+1}} \leq \frac{\frac{1}{p_n}}{\frac{1}{p_{n+1}}}$$

We can conclude the divergence of series $\sum_{n=1}^{\infty} a_n$ by using the second comparison test (theorem 1.2.9).

To prove left-to-right implication (when $\sum_{n=1}^{\infty} a_n$ is divergent), we can assume, according to the theorem 5.0.7, the existence of positive and monotonous sequence B_n such that

$$\lim_{n\to\infty} B_n = 0 \tag{6.5}$$

$$\sum_{n=1}^{\infty} a_n B_n = \infty$$

We have

$$\frac{a_n}{a_{n+1}} \leq \frac{a_n B_n}{a_{n+1} B_{n+1}} \quad \wedge \quad p_n = \frac{1}{a_n B_n} \quad \Rightarrow \quad \frac{a_n}{a_{n+1}} \leq \frac{\frac{1}{p_n}}{\frac{1}{p_{n+1}}}$$

And after some adjustments, we get the right formula

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \leq 0$$

It is not difficult to see, that $\sum \frac{1}{p_n} = \infty$ and $p_n > 0$ for all n, thus we found the numbers we were looking for. $\square$

Remark 6.1.3. Again, the requirement (6.5) is not necessary as any positive and non-increasing sequence $\{B_n\}$ with $\lim_{n\to\infty} B_n = A > 0$ will do the trick (as it is the monotony we are interested in).

As in the previous remark, if $\sum a_n$ is a divergent series and if we let $B_n = 1$ for all n then Kummer's test will confirm the divergence.

Remark 6.1.4. To sum it up, Kummer's test is very powerful because it really works for all the series with positive terms. On the other hand, using this test is equally difficult as using the first and second comparison test. The true strength of this test therefore lies in the the numbers p_n. That is, the form of this test is a masterpiece, not its contents.

To demostrate the power of Kummer's test, we show that Raabe's test and Bertrand's test are in fact its corollaries. As for Raabe's test, if we set $p_n = n$, we get

$$\exists A > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N \;:\; n\frac{a_n}{a_{n+1}} - (n+1) \geq A$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq 1 + A \qquad \text{compare with (2.3)}$$

for convergence and

$$\left(\sum \frac{1}{n} = \infty\right), \quad \exists N \in \mathbb{N}, \quad \forall n > N \;:\; n\frac{a_n}{a_{n+1}} - (n+1) \leq 0$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1 \qquad \text{compare with (2.4)}$$

for divergence.

We see, that what we can decide with Raabe's test, we can also decide with Kummer's test (with $p_n = n$) and vice versa, thus they are equivalent.

As for Kummer's version of Bertrand's test, we set $p_n = n \ln n$ and put it into the left side of (6.1):

$$n \ln n \frac{a_n}{a_{n+1}} - (n+1)\ln(n+1) =$$

$$= n \ln n \frac{a_n}{a_{n+1}} - (n+1)\left(\ln n + \ln\left(1 + \frac{1}{n}\right)\right) =$$

$$= n \ln n \frac{a_n}{a_{n+1}} - n \ln n - \ln n - \ln \left(1 + \frac{1}{n}\right)^{(n+1)} =$$
$$= \ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) - 1 - \epsilon\,(n)$$

So

$$n \ln n \frac{a_n}{a_{n+1}} - (n+1) \ln (n+1) = \ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) - 1 - \epsilon\,(n) \quad (6.6)$$

Where

$$\epsilon(n) = \ln \left(1 + \frac{1}{n}\right)^{(n+1)} - 1 = (n+1) \left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) - 1 = \frac{1}{2n} + o\left(\frac{1}{n}\right)$$

We go back to (6.1). With (6.6), if

$$\exists A > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$
$$\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \geq 1 + A + \epsilon(n) \quad (6.7)$$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Compare with Bertrand's test we worked out in the fourth chapter (see (4.4)): If

$$\exists A > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$
$$\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \geq 1 + A \quad (6.8)$$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
Since $\epsilon(n)$ can get arbitrarily small as n tends to infinity, we can hide it inside the positive constant A. Thus (6.7) and (6.8) are equivalent.

Divergence is a bit different and we will see that in this case, the tests are not equivalent. That is, Kummer's test is slightly stronger. It is because now we have zero as a sharp border, while the constant A from the previous case was quite flexible.

With (6.2), (6.6) and $p_n = n \ln n$: if

$$\left(\sum \frac{1}{n \ln n} = \infty\right), \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)-1 \le \epsilon(n)$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent. Compare with Bertrand's test (see (4.5)): If

$$\exists N \in \mathbb{N},\quad \forall n > N \;:\quad \ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)-1 \le 0$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Now let's consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{1}{n \ln n}$. Using Bertrand's test

$$\ln n\left(n\left(\frac{(n+1)\ln(n+1)}{n\ln n}-1\right)-1\right)-1 \le 0$$

$$(n+1)\ln(n+1)-n\ln n-\ln n-1 = (n+1)\ln\left(1+\frac{1}{n}\right)-1 = \epsilon(n) \le 0 \quad \rightarrow \quad false$$

Using Kummer's test (with $p_n = n \ln n$)

$$n\ln n\frac{(n+1)\ln(n+1)}{n\ln n}-(n+1)\ln(n+1) \le 0$$

$$0 \le 0 \quad \rightarrow \quad true$$

There is infinite number of series that can be decided only with Kummer's version of Bertrand's test, but if we use limits, the tests are equivalent.

Chapter 7

The Last chapter

In this chapter we will return to Raabe's test to show how we got the sequence $\frac{1}{n(\ln n)^p}$ in Bertrand's test and consequently, how to create limitless number of more and more powerful tests.

We start by analyzing when Raabe's test fails. If

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=1+\xi(n), \qquad \lim_{n\to\infty}\xi(n)=0+$$

then there exists no such constant p from (2.3), nor $1+\xi(n)\leq 1$. This means, that the given series $\sum a_n$ converges more slowly than the series $\sum \frac{1}{n^p}$, $p>1$. But it does not mean that the series $\sum a_n$ diverges. So what can we do? We need a series whose convergence/divergence is known, plus the series must converge more slowly than $\sum \frac{1}{n^p}$. A good candidate is $\sum \frac{1}{n(\ln n)^p}$, since

$$\forall\epsilon>0,\ \ \forall p>0,\ \ \exists n_0,\ \ \forall n>n_0\ :\ \ \frac{1}{n^\epsilon}<\frac{1}{(\ln n)^p}$$

(To prove use l'Hopital's rule.) So, in chapter 4, we worked out Bertrand's test from the series $\sum \frac{1}{n(\ln n)^p}$. But even this test fails when

$$\ln n\left(n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right)=1+\xi(n), \qquad \lim_{n\to\infty}\xi(n)=0+$$

and the situation repeats itself. Again, because

$$\forall\epsilon>0,\ \ \forall p>0,\ \ \exists n_0,\ \ \forall n>n_0\ :\ \ \frac{1}{(\ln n)^\epsilon}<\frac{1}{(\ln\ln n)^p}$$

we can formulate a new, more powerful criterion.

First, we have a closer look at the series

$$\sum \frac{1}{n \ln n (\ln \ln n)^p}, \quad p > 0$$

Let $f(x)$ be a function

$$f(x) = \frac{1}{x \ln x (\ln \ln x)^p}$$

on the interval (e, ∞) which satisfies $f(n) = a_n$. To use integral criteria we find the primitive function $F(x)$

$$F(x) = \int \frac{1}{x \ln x (\ln \ln x)^p dx} = \frac{\ln \ln x}{(\ln \ln x)^p} - p \int \frac{\ln \ln x}{(\ln \ln x)^{(p+1)}} \frac{1}{\ln x} \frac{1}{x} dx$$

$$F(x) = \begin{cases} \ln \ln \ln x & \text{if } \ p = 1 \\ \frac{1}{(1-p)(\ln \ln x)^{(p-1)}} & \text{if } \ p \neq 1 \end{cases}$$

Thus the series

$$\sum \frac{1}{n \ln n (\ln \ln n)^p} \begin{cases} \text{converges} & \text{if } \ p > 1 \\ \text{diverges} & \text{if } \ p \in (0, 1] \end{cases} \tag{7.1}$$

Expressing $\frac{a_n}{a_{n+1}}$

$$\frac{\frac{1}{n \ln n (\ln \ln n)^p}}{\frac{1}{(n+1) \ln (n+1) (\ln \ln (n+1))^p}} = \frac{(n+1) \ln (n+1) (\ln \ln (n+1))^p}{n \ln n (\ln \ln n)^p} =$$

(after some adjustments very similar to (4.3))

$$= 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{p}{n \ln n \ln \ln n} + o\left(\frac{1}{n \ln n \ln \ln n}\right) \tag{7.2}$$

Now we can create a new test:

Theorem 7.0.5. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. If*

$$\exists p > 1, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$

$$\ln \ln n \left(\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) - 1 \right) \geq p \tag{7.3}$$

then the series $\sum_{n=1}^{\infty} a_n$ *converges. If*

$$\exists N \in \mathbb{N}, \quad \forall n > N \ :$$

$$\ln\ln n \left(\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) - 1 \right) \leq 1 \tag{7.4}$$

then the series $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Following (7.3)

$$p > 1 \ \Rightarrow \ \exists q > 1 \ \wedge \ \exists \epsilon > 0 \ \wedge \ p = q + \epsilon$$

Thus, for sufficiently large n

$$\ln\ln n \left(\ln n \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) - 1 \right) \geq q + \epsilon \geq q + o(1)$$

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{q}{n \ln n \ln\ln n} + o\left(\frac{1}{n \ln n \ln\ln n} \right)$$

and with (7.2)

$$\frac{a_n}{a_{n+1}} \geq \frac{\frac{1}{n \ln n (\ln\ln n)^q}}{\frac{1}{(n+1) \ln (n+1) (\ln\ln (n+1))^q}}$$

Because $q > 1$ the series

$$\sum \frac{1}{n \ln n (\ln\ln n)^q}$$

converges (see (7.1)). According to the second comparison test (theorem 1.2.9) the series $\sum a_n$ converges as well.

As for the divergence, we bump here into the same problem as we did in Chapter 4 when we were proving Bertrand's test (theorem 4.1.1). (We will not point out the problem again.)

$$\frac{\frac{1}{n \ln n \ln\ln n}}{\frac{1}{(n+1) \ln (n+1) \ln\ln (n+1)}} = \frac{(n+1) \ln (n+1) \ln\ln (n+1)}{n \ln n \ln\ln n} =$$

$$= \frac{(n+1) \ln (n+1) \ln \left(\ln n + \ln \left(1 + \frac{1}{n} \right) \right)}{n \ln n \ln\ln n} =$$

$$= \frac{(n+1)\ln(n+1)\left(\ln\ln n + \ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)\right)}{n\ln n\ln\ln n} =$$

$$= \frac{(n+1)\ln(n+1)}{n\ln n} + \frac{(n+1)\ln(n+1)\ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)}{n\ln n\ln\ln n} =$$

$$= 1 + \frac{1}{n} + \frac{1}{n\ln n} + \frac{1}{n\ln n\ln\ln n} + \frac{\varsigma(n)}{n\ln n\ln\ln n}$$

Where:

$$\varsigma(n) = \ln\ln n\left((n+1)\ln\left(1+\frac{1}{n}\right) - 1\right) + (n+1)\ln(n+1)\ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right) - 1$$

With

$$\lim_{n\to\infty}\ln\ln n\left((n+1)\ln\left(1+\frac{1}{n}\right) - 1\right) = 0$$

we can simplify $\varsigma(n)$

$$\varsigma(n) \approx (n+1)\ln(n+1)\ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right) - 1 =$$

$$= (n+1)\left(\ln n + \ln\left(1+\frac{1}{n}\right)\right)\ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right) - 1$$

And pull out the biggest part

$$\varsigma(n) \approx (n+1)\ln n\ln\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right) - 1 \geq 0$$

Adjusting (7.4) we get

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n\ln n} + \frac{1}{n\ln n\ln\ln n}$$

Note that the terms we neglected in $\varsigma(n)$ were all positive so the following inequality is correct.

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{1}{n \ln n \ln \ln n} \leq$$

$$\leq 1 + \frac{1}{n} + \frac{1}{n \ln n} + \frac{1}{n \ln n \ln \ln n} + \frac{\varsigma(n)}{n \ln n \ln \ln n}$$

And finally

$$\frac{a_n}{a_{n+1}} \leq \frac{\frac{1}{n \ln n \ln \ln n}}{\frac{1}{(n+1) \ln (n+1) \ln \ln (n+1)}}$$

The second comparison test implies the divergence of the series $\sum a_n$ because the series

$$\sum \frac{1}{n \ln n \ln \ln n}$$

diverges (see (7.1)). □

Now we state one of the most general version of Bertrand's test.

Lemma 7.0.6. *Let us denote k-th ($k \in \mathbb{N}$) iteration of $n_k = \ln n_{k-1}$ ($n_0 = n$) with $\ln^k n$ (so $\ln^3 n = \ln \ln \ln n$). Then the series*

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n \ln^2 n ... \ln^{k-1} n (\ln^k n)^p} \quad p > 0 \quad k \in \mathbb{N}$$

converges for $p > 1$ and diverges for $p \leq 1$.

Proof. Let k be an arbitrary natural number. Let $f(x)$ be a function on the interval $[1, \infty)$ such that

$$f(n) = \frac{1}{n \ln n \ln^2 n ... \ln^{k-1} n (\ln^k n)^p}$$

To use integral criteria we find $F(x)$

$$F(x) = \int \frac{1}{x \ln x \ln^2 x ... \ln^{k-1} x (\ln^k x)^p} dx =$$

$$= \frac{ln^k x}{(\ln^k x)^p} - p \int \frac{ln^k x}{(\ln^k x)^{p+1} \ln^{k-1} x ... (\ln x) x} dx$$

$$F(x) = \begin{cases} \ln^k x & \text{if } \ p = 1 \\ \frac{1}{(1-p)(\ln^k x)^{p-1}} & \text{if } \ p \neq 1 \end{cases}$$

Now we examine $\lim_{x\to\infty} F(x)$ and we can conclude that the series

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n \ln^2 n ... \ln^{k-1} n (\ln^k n)^p} \begin{cases} \text{converges} & \text{if } \ p > 1 \\ \text{diverges} & \text{if } \ p \in (0, 1] \end{cases}$$

□

Lemma 7.0.7. *Using the notation from the previous lemma*

$$\forall \epsilon > 0, \quad \forall p \in \mathbb{N}, \quad \forall k \in \mathbb{N}, \quad \exists n_0 \quad : \quad \forall n > n_0 \quad \frac{1}{(\ln^k n)^\epsilon} < \frac{1}{(\ln^{k+1} n)^p}$$

Proof. Use substitution $m = \ln^k n$ and l'Hopital's rule. □

Lemma 7.0.8.

$$\frac{(n+1)\ln(n+1)\ln^2(n+1)\ldots\ln^k(n+1)}{n \ln n \ln^2 n \ldots \ln^k n} =$$

$$= 1 + \frac{1}{n} + \frac{1}{n \ln n} + \ldots + \frac{1}{n \ln n ... ln^k n} + \frac{\vartheta(n)}{n \ln n ... ln^k n}$$

where

$$\vartheta(n) \geq 0 \quad \textit{and} \quad \lim_{n\to\infty} \vartheta(n) = 0$$

Theorem 7.0.9. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive memebers. If*

$$\exists k \in \mathbb{N}, \quad \exists p > 1, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$

$$\ln^k n \Big(\ln^{k-1} n \Big(... \ln n \big(n (\frac{a_n}{a_{n+1}} - 1) - 1 \big) ... - 1 \Big) - 1 \Big) \geq p$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\exists k \in \mathbb{N}, \quad \exists N \in \mathbb{N}, \quad \forall n > N \ :$$

$$\ln^k n \Big(\ln^{k-1} n \Big(... \ln n \big(n (\frac{a_n}{a_{n+1}} - 1) - 1 \big) ... - 1 \Big) - 1 \Big) \leq 1$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Again, the theorem can be made into equivalence. We will use

$$\prod_{i=1}^{k} \ln^i n = (\ln n)(\ln^2 n)...(\ln^k n)$$

Theorem 7.0.10. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive memebers. Assume that there exist a natural number k, a real number p, a number $r > 1$ and a real bounded sequence B_n and*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \sum_{j=1}^{k-1} \frac{1}{n \prod_{i=1}^{j} \ln^i n} + \frac{p}{n \prod_{i=1}^{k} \ln^i n} + \frac{B_n}{n\left(\prod_{i=1}^{k-1} \ln^i n\right)(\ln^k n)^r}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $p > 1$. And the series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $p \leq 1$.

Even though it might seem that this is a powerful tool to test convergence, it actually can resolve only a fraction of all possible series (with positive terms). We will discuss the results in the chapter named Epilogue.

An example of the case when the upper criterion fails. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonous positive sequence and $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence such that

$$b_n = \frac{1}{n} \quad \Leftrightarrow \quad n = a^2, \quad a \in \mathbb{N}$$

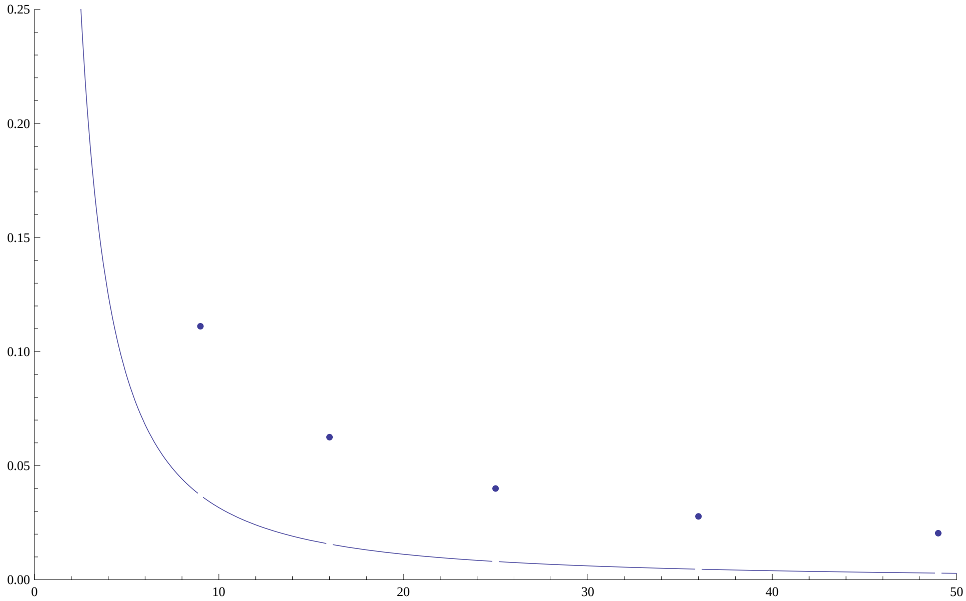
$$b_n = 0 \text{ for all other } n$$

It is easy to see that the series $\sum_{n=1}^{\infty} b_n$ converges (moreover, if we look at the index n and the term a_n, it converges very slowly). Now we use these two series to create new series $\sum_{n=1}^{\infty} \Xi_n$:

$$\Xi_n = a_n \quad \Leftrightarrow \quad b_n = 0$$

$$\Xi_n = b_n \quad \Leftrightarrow \quad b_n \neq 0$$

$$\{\Xi_n\}_{n=1}^{\infty} \ :$$

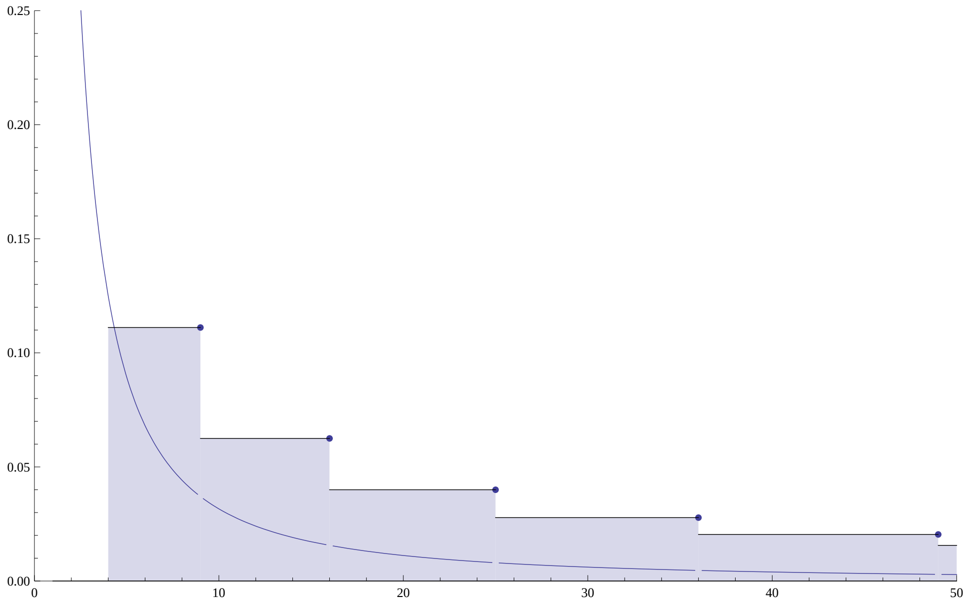


Obviously, the series $\sum_{n=1}^{\infty} \Xi_n$ converges. Now let's try to find such least monotonous sequence $\{\Upsilon_n\}_{n=1}^{\infty}$ that

$$\forall n \ : \ \ \Xi_n \leq \Upsilon_n$$

It is not very difficult to see that the sequence we are looking for is

$$\Upsilon_k = \frac{1}{n^2} \quad \Leftrightarrow \quad k \in ((n-1)^2, n^2], \quad n \in \mathbb{N}$$



Moreover, there are exactly $(2n-1)$ terms of value $\frac{1}{n^2}$ in the sequence $\{\Upsilon_k\}_{k=1}^{\infty}$. Therefore

$$\sum_{k=1}^{\infty} \Upsilon_k = \sum_{n=1}^{\infty} \frac{2n-1}{n^2} \approx \sum_{n=1}^{\infty} \frac{1}{n} = \infty \tag{7.5}$$

Sequence $\{\Upsilon_n\}_{n=1}^{\infty}$ is the least monotonous bound of sequence $\{\Xi_n\}_{n=1}^{\infty}$ and because the convergent series from the theorem 7.0.9 is also monotonous, it cannot be an upper bound for this sequence (otherwise it would bound a divergent series from (7.5)).

Chapter 8

Epilogue

And that's all folks. We hope that this work has helped all readers to familiarize with the techniques used to determine convergence/divergence of infinite series (with positive terms only, though). But also to realize, that these special tests (Raabe, Gauss, Bertrand) will work only for a small fraction of them and that no given series can lead to a universal comparison test. The example from the previous chapter can also serve as a proof that no monotonous series can lead to a test that will be able to decide all positive series. We are sure that the reader can see where the problem arises.

Note that we are distinguishing between the terms "universal criterion" and "general criterion". The former means a test, based on a fixed series, that can unravel the character of all series with positive terms. That is, the usage of such criterion is straithforward. The latter means a test (e.g. first or second comparison test) that works for all positive series, but it usually requires additional (and often big) effort to make a good use of it (as a price for its generality).

And that is the situation with Kummer's test, as this is a general test and it truly works for all the series (with positive terms) there are. However, when looked at more closely, we can see that it is just the first and second comparison criterion in a smart disguise. So making this test work (finding the numbers p_n) can be very difficult. On the other hand, the benefit of this test is that we can see many special criterions as its corollaries. That is, the treasure is the treasure chest itself, not its contents, and this is probably the best result we can expect to get with general criterions.

Last but not least, thank you for reading this work. We hope that someday, somewhere, it will come in hand.

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