Commenius University
Faculty of Mathematics, Physics and Informatics
Institute of Informatics

Flows In Cayley Graphs

(Master's Thesis)

Mária Nánásiová

Adviser: doc. RNDr. Martin Škoviera, PhD.

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By this I declare that I wrote this thesis by oneself, only with the help of the referenced literature, under the careful supervision of my thesis adviser.

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Contents

1 Introduction

The notion of a flow on a graph belongs to important concepts of contemporary graph theory with many applications both in graph theory and beyond. Specifically a nowhere-zero k-flow on a graph G is an orientation of G and an assignment of values $1, ..., k-1$ to the directed edges of G in such a way that the Kirchhoff Current Law is satisfied, that is, the sum of inflowing values equals the sum of outflowing values. A nowhere-zero A-flow, where A is any Abelian group, is defined simmilary. In this case, the flow values are non-zero elements of A.

A systematic study of nowhere-zero flows begins with the seminal papers [17] and [18] of Tutte where, among others, are proposed the following three conjectures.

5-*Flow Conjecture:* Every bridgeless graph has a nowhere-zero 5-flow.

4-*Flow Conjecture:* Every bridgeless graph with no Petersen minor admits a nowhere-zero 4-flow.

3-*Flow Conjecture:* Every bridgeless graph without 3-edge-cuts admits a nowhere-zero 3-flow.

It is obvious that a graph with a bridge cannot have a nowhere-zero k-flow for any k. On the other hand, it is not obvious whether there exists a finite "universal" bound n such that every bridgeless graph admits a nowhere-zero n-flow. The existence of such a bound was independently established by Kilpatrick and Jaeger [10], [9]. They proved that every bridgeless graph has a nowhere-zero 8-flow. This result was later superseded by Seymour [15] who showed that every bridgeless graph has a nowhere-zero 6-flow.

The question whether $k = 5$ satisfies the condition in the conjecture remains unanswered. Still, the bound 5 is best possible since the Petersen graph is bridgeless and has no nowhere-zero 4-flow.

Even if correct, the 4-Flow Conjecture will not be the best possible as every complete graph with at least 10 vertices contains the Petersen graph as a minor and has a nowhere-zero 3-flow, see [5] page 134.

Cubic bridgeless graphs without a 4-flow are called snarks. Therefore, the 4-Flow Conjecture for cubic graphs states that every snark contains the Petersen graph as a minor.

Furthemore, Jaeger showed that every 4-edge-connected graph admits a nowhere-zero 4-flow. Note that any vertex-transitive graph of valency k is k-edge-connected. It follows that every vertex-transitive graph of valency 4 has a nowhere-zero 4-flow. As Cayley graphs are vertex-transitive, Cayley graphs of valency at least four admit a nowhere-zero 4-flow.

Tutte proved that there is a strong connection between k-flows and A-flows. A graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for an Abelian group A. Since the sum of two A-flows is again an A-flow, it is sometimes easier to construct A-flows than k-flows. Naturally, the same does not apply for nowhere-zero A-flows.

Another theorem states that if a graph G has a nowhere-zero flow with at most k different values, it also has a $\mathbb{Z}_{k+1}$-flow. For $k \geq 5$, this is a trivial application of Seymour's 6-Flow Theorem. When $k \leq 4$, the proof is based on a number theory problem called "Lonely Runner Conjecture".

Another well known conjecture is that any Cayley graph of valency at least four has a Hamilton cycle. It follows from the connection between $\mathbb{Z}_k$-flows and k-flows that any graph with a Hamiltonian cycle admits a nowhere-zero 4-flow. Unfortunately, we cannot deduce that Hamiltonian graphs have a nowhere-zero 3-flow.

In this context we have to mention Babai's [2] counterconjecture saying that not only there exists a Cayley graph which is non-Hamiltonian, but for some constant $c \geq 0$, there are infinitely many vertex-transitive graphs, even Cayley graphs, without cycles of length greater than $(1 - c)n$, n being the order of the graph. In particular, there exists a non-Hamiltonian arbitrary large Cayley graph.

Nowhere-zero k-flows are closely related to the existence of k-colourings of graphs embedded in orientable surfaces. Tutte [17] showed that a graph embedded in an orientable surface S has a vertex k-colouring if and only if its dual on S has a nowhere-zero k-flow. In particular, the Four Colour Theorem is equivalent to the assertion that every bridgeless cubic planar graph has a nowhere-zero 4-flow, in other words, there is no planar snark. A related theorem by Grötzsch [6] states that every bridgeless planar graph with no vertices of valency three admits a nowhere-zero 3-flow.

From here there is a short way to flows on Cayley graphs. As we mentioned before, any 4-edge-connected graph admits a nowhere-zero 4-flow. Since Cayley graphs are vertex transitive, Cayley graphs of valency at least four admit a nowhere-zero 4-flow.

In 1996, Alspach, Liu and Zhang [1] proved that every Cayley graph of a solvable group of order greater than 2 admits a nowhere-zero 4-flow. The crucial cubic case of their result was improved by Nedela and Škoviera [11] extending it to a broader class of Cayley graphs. They proved, that if there exists a Cayley snark, then there is one on a simple or an almost simle non

Abelian group. Another improvement is due to Potočnik [12]. He generalized the result to the graphs which admit a vertex-transitive action of a solvable group (excluding the Petersen graph).

Finally, Potočnik, Škoviera and Škrekovski showed that every Cayley graph of valency at least four on an Abelian group has a nowhere-zero 3-flow in [13].

The main purpose of the present work is to prove the following two theorems.

Theorem A *Every Cayley graph* $\mathrm{Cay}(G, S)$ *on nilpotent group of valency at least four has a nowhere-zero* 3*-flow.*

Theorem B *Let* G *be a group containing an Abelian subgroup of index two. Then every Cayley graph* $\mathrm{Cay}(G, S)$ *of valency at least four has a nowhere-zero* 3*-flow.*

2 Preliminaries

The *graph* is a quadruple $G = (V, D, L, I)$ where $D = D(G)$ and $V = V(G)$ are disjoint non-empty finite sets, $I \to V$ is a surjective mapping, and L is an involutory permutation on D. The elements of D and V are *darts* and *vertices*, respectively, I is the incidence function assigning to every dart its *initial vertex* and L is the *dart-reversing* function where $L(x) = x^{-1}$. The orbits of the group $\langle L \rangle$ on D are *edges* of K. We do not allow $L(x) = x$.

The valency of a vertex v, written as $val(v)$, is the number of edges incident to v. Graphs where all vertices have equal valencies are called *regular* graphs.

Let A be an abelian group with additive notation. A function $f\colon D(X) \to A$ is an *A-flow* on X if the following two conditions are satisfied:

(i) $f(x^{-1}) = -f(x)$, for each dart $x \in D(X)$;

(ii) $\sum_{x \in D(u)} f(x) = 0$, for each vertex $u \in V(X)$.

If $f(x) \neq 0$ for each dart x, f is called a *nowhere-zero A-flow* on X. A $\mathbb{Z}$-flow which takes values in $\{1, \dots, (k-1)\}$ is called a *nowhere-zero k-flow*.

The concept of a Cayley graph was first introduced in 1878 by Cayley in [4]. Cayley graphs are useful in interconnection network theory (see [7] for references). Given a group G with an identity element 1 and a sequence $S = \{s_1, s_2, \ldots, s_n\}$ of elements of $G - \{1\}$ such that $S^{-1} = S$, the darts of $\mathrm{Cay}(G, S)$ are ordered pairs (g, s_i) where $g \in G$ and $s_i \in S$. The dart (g, s_i) has initial vertex g and terminal vertex gs_i, and $L(g, s_i) = (gs_i, s_i^{-1})$. The sequence S is called Cayley sequence.

Note that such a graph is connected if and only if elements of S generate G. Nevertheless, in this paper we allow Cayley graphs to be disconnected.

Cayley graphs are highly symmetric. They are vertex transitive and therefore regular. As we mentioned before, each Cayley graph of valency at least four admits a nowhere-zero 4-flow. Also, all Cayley graphs on solvable groups admit nowhere-zero 4-flows.

General problems related to search for nowhere-zero 3-flows are difficult and many of them are left open for years. Rather then attempting to solve them, we will concentrate on special classes of groups with understandable and comfortable structure. As the fact that Cayley graphs on Abelian groups (of valency at least 4) have a nowhere-zero 3-flow is already proved, we will focus on group classes with an Abelian normal subgroup. We will also show some results on groups with non-empty centre.

It is assumed that the reader is familial with the fundamentals of group theory. More details can be found for example in [14]. All groups in this work are assumed to be finite.

Recall that a subgroup H of a group G is *normal*, denoted by $H \trianglelefteq G$, if $gHg^{-1} \subseteq H$ for each $g \in G$. That is, if H is invariant under all *inner* automorphisms of G. If H is invariant under *all* automorphisms, then H is called a *characteristic* subgroup of G.

Lemma 2.1 *Let G be a group with a normal subgroup H. If $K \trianglelefteq G$ and $K \subseteq H$, then $K \trianglelefteq H$ and $H/K \trianglelefteq G/K$.*

The *centre* of a group G, denoted $Z(G)$, is a subset of all elements such that $gh = hg$ for each $h \in G$. The centre plays an important role in the study of a group structure. Note that it is a characteristic subgroup of G.

Among groups with a non-empty centre, an important role is played by *p-groups.* For any prime number p, a p-group is a group of order being a power of p. Since the order of a subgroup divides the order of the group, any subgroup of a p-group is also a p-group. It follows, that any quotient group of a p-group is a p-group, too.

Many structural properties of finite groups depend on p-subgroups of a given graph. Given a group G, a p-Sylow subgroup of G is any p-group which is maximal with respect to inclusion.

A group G is called *nilpotent* if it has a normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

such that $G_{j+1}/G_j \leq Z(G/G_j)$ for any j. Such a normal series is called a *central series* of G. The shortest length of nilpotent series of a group is called the *nilpotency* class of G. The following theorem is due to Burnside and Wieland [14].

Theorem 2.1 *A nilpotent group is isomorphic to a direct product of its Sylow subgroups. In particular, every nilpotent group is isomorphic to a direct product of p-groups.*

Nilpotent groups have similar properties to p-groups. They have a non-empty centre and any subgroup of a nilpotent group is nilpotent. Moreover, any quotient group of nilpotent group is nilpotent too.

Theorem 2.2 *The class of nilpotent groups is closed under taking subgroup, quotiens and direct products.*

Finally we define solvable groups, a class of groups that contains all nilpotent and Abelian groups.

A group G is called *solvable* if it has a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq ... \trianglerighteq G_{n-1} \trianglerighteq G_n = \{1\}$$

with Abelian factor group G_{i+1}/G_i for i in $\{1, ..., n\}$. Such a normal series is called a *solvable series* of G.

There is an alternative approach to solvable groups which is longer, however, makes some useful properties easier to deduce. First, we introduce the term commutator.

Let G be a group and $g, h \in G$. The *commutator* of g and h is

$$[g, h] = ghg^{-1}h^{-1}.$$

The *commutator subgroup* of G, denoted $G^{'}$, $G^{(1)}$ or $[G, G]$, is the subgroup of G generated by commutators of its elements. Such a subgroup is also called the *derived subgroup* of G.

Inductively, we can define $G^{(2)}$ as the commutator subgroup of $G^{(1)}$, $G^{(3)}$ as the commutator subgroup of $G^{(2)}$, etc. The subgroup $G^{(i)}$ is the *i-th derived* subgroup of G.

The commutator subgroup of the group has some interesting properties. First of all, $[g, h] = e$ if and only if g and h commute. Therefore, G is Abelian if and only if $G^{(1)}$ is trivial. In general, the subgroup $G^{'}$ measures the commutativity of a group. The smaller $G^{'}$, the "more commutative" it is.

Let α be an automorphism of the group G. Note that

$$\alpha([g, h]) = \alpha(ghg^{-1}h^{-1}) = \alpha(g)\alpha(h)\alpha(g^{-1})\alpha(h^{-1}) = [\alpha(g), \alpha(h)]$$

Therefore, $\alpha(G^{'}) = G^{'}$ for any automorphism α. It is easy to see that $G^{(m)}$ is a characteristic subgroup of the group G for any integer m.

Now, we are ready to introduce alternative description of solvable groups.

Theorem 2.3 *A group G is solvable if and only if for some $G^{(k)}$ is trivial $k \in \mathbb{N}$.*

As a consequence of this theorem, any solvable group has an Abelian characteristic subgroup $G^{(k-1)}$. The following theorem is the one that allows us to use induction in later proofs.

Theorem 2.4 *The class of all solvable groups is closed under taking subgroups, quotient groups and direct products.*

We will finish this part by folloving theorem concerning group theory. The proof can be found in [14].

Theorem 2.5 *Every Abelian group is isomorphic to a direct product of cyclic groups.*

3 3-Flows In Cayley Graphs On Abelian Groups

By a *ladder* we mean a graph isomorphic to $P_n \times K_2$ where $V(P_n) = \{1, ..., n\}$ and $V(K_2) = \{1, 2\}$. Adding the edges $11, 2n$ and $1n, 21$ to the ladder a *Moebius ladder* M_n is obtained. Such a graph is bipartite if and only if n is odd. A *circular ladder* $C_n \times K_2$ can be obtained by joining the vertices 11,1n and 21 and 2n. Unlike the Moebius ladder M_n, the circular ladder is bipartite if and only if n is even. We refer to any graph homeomorhic to one of these two graphs as a *cyclic ladder*.

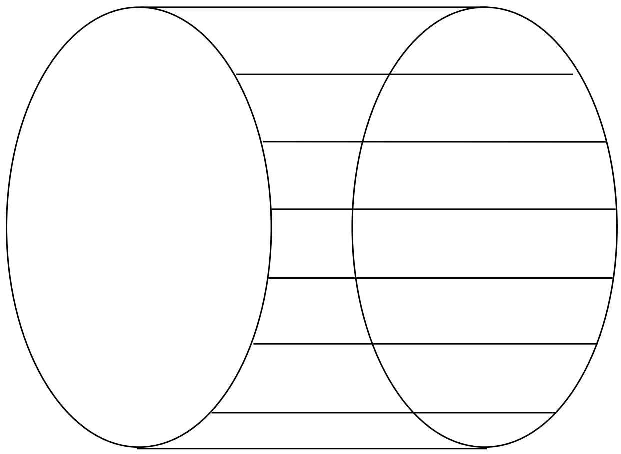

Figure 1. Circulant ladder with eight rungs. It is bipartite.

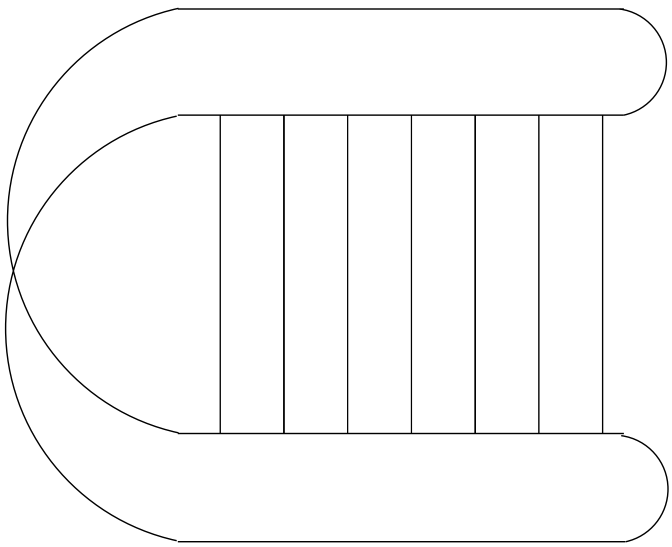

Figure 2. Moebius ladder with seven rungs. It is also bipartite and therefore admits a nowhere-zero 3-flow.

Lemma 3.1 *Let* $\mathrm{Cay}(G,S)$ *be a connected cubic Cayley graph, where* S *contains a central involution of* G*. Then* G *is isomorphic to one of following groups:* $\mathbb{Z}_{2n}$*,* $\mathbb{Z}_n \times \mathbb{Z}_2$*,* $\mathbb{D}_{2n}$*,* $\mathbb{D}_n \times \mathbb{Z}_2$*,* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$*. In any case,* $\mathrm{Cay}(G,S)$ *is cyclic ladder.*

Proof. Let c be a central involution and let $S = \{x, x^{-1}, c\}$. If the involution $c \in \langle x \rangle$, then clearly $G = \mathbb{Z}_{2n}$. Otherwise, if $c \notin \langle x \rangle$, then $G = \mathbb{Z}_n \times \mathbb{Z}_2$.

Let all elements of S be involutions x, y, c. In the case of $xy = yx$ we have $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or, if $xy = c$, then $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Otherwise, if the group $\langle x, y \rangle$ contains c, then $G = \mathbb{D}_{2n}$. If c is not member of $\langle x, y \rangle$, then $G = \mathbb{D}_n \times \mathbb{Z}_2$. □

Lemma 3.2 *Every cyclic ladder has a nowhere-zero* 3*-flow or has a* 3*-flow, where the zero value occurs on a single arbitrary rung.*

Proof. We can easily see that any cyclic ladder $C_n \times K_2$ with one rung removed is homeomorphic to the cyclic ladder $C_{n-1} \times K_2$. Therefore, if n is odd and the graph $C_n \times K_2$ does not admit a nowhere-zero 3-flow, then $C_{n-1} \times K_2$ has a nowhere-zero 3-flow.

Analogously, a Moebius ladder M_n without one rung is homeomorphic to another Moebius ladder M_{n-1}. If the graph M_n does not admit a nowhere-zero 3-flow, then we can find a nowhere-zero 3-flow of the Moebius ladder M_{n-1} which completes the proof. □

A technique used in [13] consists of decomposing a given graph into several edge-disjoint subgraphs, each having a nowhere-zero 3-flow. Usually the graph is divided into a cubic bipartite spanning subgraph and a subgraph of even valency. A cubic bipartite graph admits a nowhere-zero 3-flow and a graph of even valency has a nowhere-zero 2-flow, therefore this decomposition gives us a nowhere-zero 3-flow on the original graph.

Any generating set of Cayley graph of valency at least seven contains two involutions or a non-involution. Edges generated by theses elements form a spanning subgraph of valency two. What we obtain by deleting them is a smaller Cayley graph. Therefore, to prove that a nowhere-zero 3-flow exists on Cayley graphs of valency at least five, it is sufficent to show that there exists a nowhere-zero 3-flow on Cayley graphs of valency five.

Theorem 3.1 *Let* $\mathrm{Cay}(G,S)$ *be a Cayley graph of valency at least four such that* S *contains a central involution. Then* $\mathrm{Cay}(G,S)$ *has a nowhere-zero 3-flow.*

Proof. Clearly, if the valency of $\mathrm{Cay}(G,S)$ has a nowhere-zero 3-flow.

It is always possible to find two Cayley subsequences S_1, S_2 of the sequence S with intersection containing only central involution c and both of them having exactly three elements. Then $S' = S \setminus (S_1 \cup S_2)$ is a Cayley sequence of even length.

If follows from Lemma 3.1 that both $\mathrm{Cay}(G,S_1)$ and $\mathrm{Cay}(G,S_2)$ are cubic and composed of cyclic ladders. Any c-edge is contained in exactly two different cyclic ladders. If at least one of these cyclic ladders, say S_1, has a nowhere-zero 3-flow, then $\mathrm{Cay}(G,S)$ can be decomposed into a cubic bipartite graph $\mathrm{Cay}(G,S_1)$ and a graph of even valency $\mathrm{Cay}(G,S\setminus S_1)$ and therefore $\mathrm{Cay}(G,S)$ has a nowhere-zero 3-flow.

Otherwise, we have to build the flow inductively. Let us start with the set T_1 containing an arbitrary cyclic ladder. It follows from Lemma 3.2, that it has 3-flow f such that there exists exactly one c-edge with flow value zero.

In general, let T_i be the set of cyclic ladders and f_i 3-flow on $\mathrm{Cay}(G,S)$ satisfying:

- If there is an edge uv in T_i such that $f_i(uv) = 0$, then the edge is a c-edge.
- The flow $f_i(uv)$ is non zero for all c-edges uv contained in two different ladders from T_i.
- For any edge that is not contained in T_i, the flow f_i is zero.
- There is no more than one edge (c-edge) from T_i with flow value of 0.

It is easy to see, that T_1 has these properties. Now we define T_{i+1}. If the set T_i does not contain an edge with flow value zero, we can pick any unused cyclic ladder L and set T_{i+1} as $T_i \cup L$. According to Lemma 3.2, it is possible to find a flow l on L which is a nowhere-zero 3-flow or 3-flow where the zero value occurs on a single rung. Then, we can define $f_{i+1} = f_i + l$. Since there is no edge with both flow values f_i and l non-zero, f_{i+1} satisfies conditions mentioned above.

Otherwise, let uv be a c-edge with flow value zero. It follows from the second condition, that there is a circular ladder L_2 not in T_i containing the edge uv. What we obtain by deleting all rungs from the intersection of L_2 and T_i except uv from the circular ladder L_2 is a new circular ladder L_1.

Consider the case where L_1 has more than one rung or is bipartite. Again, we use Lemma 3.2 to find a flow $f^{'}$ on L_1 such that the flow value on the edge uv is not zero. Now, we can put $T_{i+1} = T_i \cup L_1$ and define f_{i+1} as the sum of flows $f^{'}$ and f_i. Thus, f_{i+1} satisfies conditions mentioned above.

Finally, if L_1 has exactly one rung and is not bipartite, then we can add one more rung gh from $T_i \cap L_1$ into L_1 and find a nowhere-zero 3-flow $f^{''}$ on it. Now, the intersection of L_1 and T_i contains exactly two c-edges.

Let us inspect the flows sum $f_i + f^{''}$. The only edge where both flows f_i, $f^{''}$ are non-zero is the rung gh. If $f_i(gh) + f^{''}(gh) \neq 0$, then the flow f_{i+1} will be set as $f_i + f^{''}$. Otherwise, f_{i+1} will be $f_i(gh) - f^{''}(gh)$. Clearly, f_{i+1} is a nowhere-zero 3-flow on T_{i+1}

As $\mathrm{Cay}(G, S_1 \cup S_2)$ is a finite graph, there is an index k such that T_k covers whole graph and f_k is a nowhere-zero 3-flow on $\mathrm{Cay}(G, S_1 \cup S_2)$. Since $\mathrm{Cay}(G, S) = \mathrm{Cay}(G, S_1 \cup S_2) \cup \mathrm{Cay}(G, S^{'})$ and $\mathrm{Cay}(G, S^{'})$ is of even valency, $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow. □

The following theorem was proved in [13], however we provide shorter proof.

Theorem 3.2 *Every Cayley graph* $\mathrm{Cay}(G, S)$ *on Abelian group of valency at least four has a nowhere-zero* 3*-flow.*

Proof. If $\mathrm{Cay}(G, S)$ is of even valency then $\mathrm{Cay}(G, S)$ admits a nowhere-zero 3-flow. In the case of the graph of odd valency, there is is a central involution in the sequence S. Therefore, the theorem is a straiforward consequence of Theorem 3.1. □

4 3-Flows In Cayley Graphs On Nilpotent Groups

Lemma 4.1 *Let G be a group and H a normal subgroup of G. Let S be a Cayley sequence with empty intersection with H. If $\mathrm{Cay}(G/H, S/H)$ has a nowhere-zero 3-flow, then $\mathrm{Cay}(G, S)$ has also a nowhere-zero 3-flow.*

Proof. Let f_1 be a nowhere-zero 3-flow on $\mathrm{Cay}(G/H, S/H)$. For every edge xy from the graph $\mathrm{Cay}(G, S)$, there is one corresponding edge $xHyH$ from the graph $\mathrm{Cay}(G/H, S/H))$. Let us define f_2 by setting

$$f_2(xy) = f_1(xHyH)$$

Clearly, f_2 is a nowhere-zero 3-flow on $\mathrm{Cay}(G, S)$. □

Lemma 4.2 *Let G be a finite group and let $H \trianglelefteq G$ be a normal Abelian subgroup of even order. Then G contains a central involution.*

Proof. Since H is Abelian, there exist cyclic subgroups $H_1 = \langle b_1 \rangle, H_2 = \langle b_2 \rangle, ..., H_r = \langle b_r \rangle$ such that $H_i \cap H_j = 1$ for $i \neq j$ and $H = H_1 H_2 ... H_r$. We may assume that for $i = 1, 2, ..., s$, $|H_i|$ is even, and for $i \geq s+1$, $|H_i|$ is odd. Define $a = a_1 a_2 ... a_s$ where each a_i is the only non-trivial involution of H_i. Since H is Abelian, a is a fixed by each automorphism of H. In particular, a is fixed by every inner automorphism, and hence $a \in Z(G)$ □

Theorem 4.1 *Let G be a group with an Abelian normal subgroup H. Let $\mathrm{Cay}(G, S)$ be a Cayley graph of valency at least than four such that there is an involution in the $S \cap H$. Then $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow.*

Proof. We prove Lemma by induction on $|H|$. In the case of $|H|$=2, $c \in H \cap S$ is a central involution and $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow as a consequence of Theorem 3.1. Therefore we may use it to start the induction.

Since H contains an involution, it has even order. It follows from Lemma 4.2, that there is a central involution c in H.

We will consider two different possibilities. If the involution c belongs to the Cayley sequence S, then we can use Theorem 3.1 to find a nowhere-zero 3-flow.

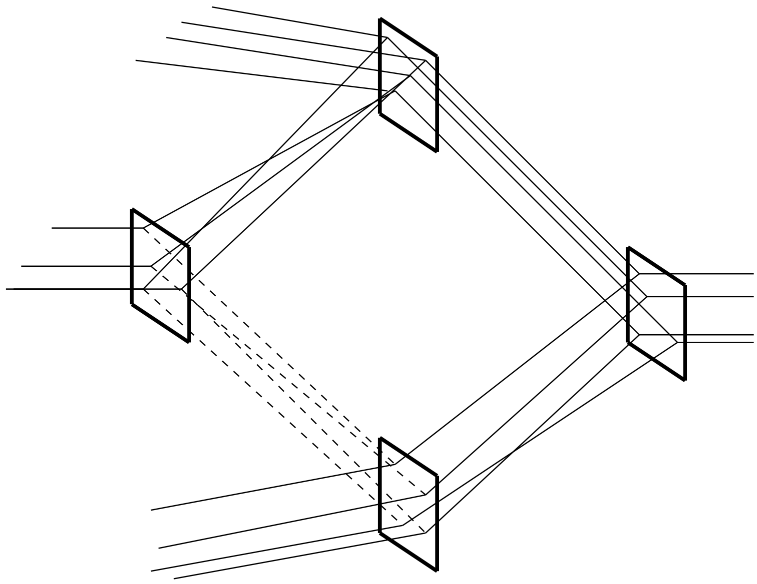

Figure 3. Segment of a Cayley graph on a group G with a normal subgroup H. Squares reprezents cosets of H. No generator belongs to the subgroup H.

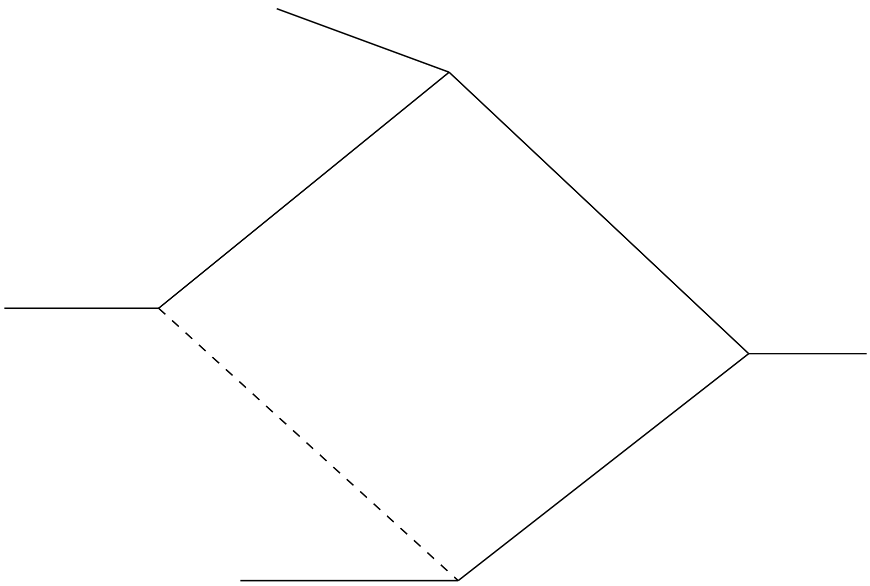

Figure 4. Segment of a factor Cayley graph on a factor group G/H. The dotted line represents corresponding edge to the dotted edges in the original graph.

If the involution c is not member of the sequence S, then $H_1 = \langle c \rangle$ is normal subgroup of G and H with empty intersection with S. Since

$$gH_1hH_1g^{-1}H_1 = ghg^{-1}H_1 \in H/H_1,$$

H/H_1 is normal Abelian subgoup of G/H_1 and $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow as a result of the Lemma 4.1 and induction hypothesis. □

As all solvable groups admit Abelian normal subgroup, a Cayley graph on a solvable group with a generating involution in the Abelian normal subgroup of valency at least four has a nowhere-zero 3-flow.

Theorem 4.2 *Every Cayley graph* $\mathrm{Cay}(G, S)$ *on nilpotent group of valency at least four has a nowhere-zero* 3*-flow.*

Proof. If $\mathrm{Cay}(G, S)$ is of even valency, then $\mathrm{Cay}(G, S)$ is Eulerian and has a nowhere-zero 3-flow.

If the graph $\mathrm{Cay}(G, S)$ is of odd valency, we prove Lemma by induction on $|G|$. Any cyclic group of prime order is Abelian and nilpotent. As we proved before in the Theorem 3.2, such a group admits a nowhere-zero 3-flow. Therefore we may use it as induction start.

Since G contains an involution, its 2-Sylow subgroup is non-trivial and G has a centre of even order. Therefore, G has a central involution c.

We will consider two different possibilities. If the involution c belongs to the sequence S, then we can use Theorem 3.1 to find a nowhere-zero 3-flow. If the involution c is not a member of the sequence S, then $\langle c \rangle$ is a normal subgroup of G with empty intersection with S, and $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow as a result of Lemma 4.1 and of induction hypothesis. □

5 Additional results

Lemma 5.1 *A Cayley graph on an Abelian group generated by four involutions has a cubic bipartite spanning Cayley subgraph.*

Proof. Let a, b, c, d be generating involutions. First, consider Cayley subgraph G generated by elements a, b, c. If $a = b = c$, then G is clearly cubic bipartite. Without loss of generality, let $a \neq b$. If $ab \neq c$, then $\mathrm{Cay}(G, \{a, b, c\})$ is isomorphic to a vertex disjoint union of cubes (or circular ladder with four rungs) and therefore is bipartite. The case of $ab \neq d$ is analogous. Finally, let $c = d = ab$ and consider a graph generated by elements a, c, d. Such a graph is isomorphic to a circular ladder with two rungs which is cubic bipartite. □

Note that involutions a, b, c, d in the previous Lemma do not have to be distinct. Also, Cayley subgraph obtained in the previous Lemma admits a nowhere-zero 3-flow.

Lemma 5.2 *Let G be a group and let H be a normal Abelian subgroup of index two. Let $H_1 \trianglelefteq G$ such that $H_1 \trianglelefteq H$. In such a situation, $H' = H/H_1$ is normal Abelian subgroup of $G' = G/H_1$ of index two.*

Proof. Let g be an element from $G \setminus H$ and h be an element from H. Clearly, gH_1 is not in H' and $\langle H, g\rangle = G$. Factor group of Abelian group is Abelian, so H' is Abelian. Since H_1 is characteristic subgroup of H, then $gH_1 = H_1g$ and $g^{-1}H_1 = H_1g^{-1}$. Therefore,

$$gH_1hH_1g^{-1}H_1 = ghg^{-1}H_1 \in H'$$

and H' is normal Abelian subgroup of G'.

Finally, we have to prove that $|G'/H'| = 2$. From $\langle H, g\rangle = G$, see that $H' \cup gH_1 = G'$ and therefore $|G'/H'| = 2$. □

Lemma 5.3 *Let G be a group with an involutory element c and non-involutory element d. If there exists an integer k such that $d(cd)^k = 1$, then c and d commute.*

Proof.

$$d(cd)^k = 1$$
$$cd(cd)^k c = c1c$$
$$(cd)^{k+1} c = 1$$

Now, let us solve folloving equation:

$$dc = cx$$
$$cdc = x$$
$$d(cd)^k cdc = x$$
$$d(cd)^{k+1} c = x$$
$$d = x$$

Therefore $dc = cd$. □

Lemma 5.4 *Let* $\mathrm{Cay}(G, \{c, d\})$ *be a Cayley graph on an Abelian group. If there is an alternating* (c, d)*-cycle, then there is a spanning set of alternating vertex disjoint* (c, d)*-cycles.*

Proof. Let T_1 be a set containing only an alternating cycle (c, d)-cycle passing throught vertex 1. If all elements of G are contained in T_1, then T_1 is a spanning set of alternating vertex disjoint (c, d)-cycles.

Let T_i be a set containing i alternating vertex disjoin (c, d)-cycles such that for any two cycles $v_1, v_2, ..., v_k$ and $u_1, u_2, ..., u_k$ in T_i exists an element from G satisfying $v_j = hu_j$; $j \in \langle 1, k \rangle$

We define T_{i+1} as follows. Let $v_1, v_2, ..., v_k$ be a cycle in T_i. Clearly, if the set T_i is not spanning, then there exists $h \in G$ such that $hv_j \notin T_i$ for any j. Then $hv_1, hv_2, ..., hv_k$ is an alternating (c, d)-cycle and $T_{i+1} = T_i \cup hv_1, hv_2, ..., hv_k$ is a set containing $i + 1$ alternating vertex disjoin (c, d)-cycles.

As G is finite, there is an index k such that T_k covers all vertices of G by vertex disjoint (c, d)-cycles. □

Theorem 5.1 *Let* $\mathrm{Cay}(G, S)$ *be a cubic Cayley graph where* G *has a normal Abelian subgroup* H *of index two. If the intersection of* S *and* H *does not contain an involution,* $|S \cap H| = 2$ *and the sequence* S *is not Abelian, then* $\mathrm{Cay}(G, S)$ *is bipartite.*

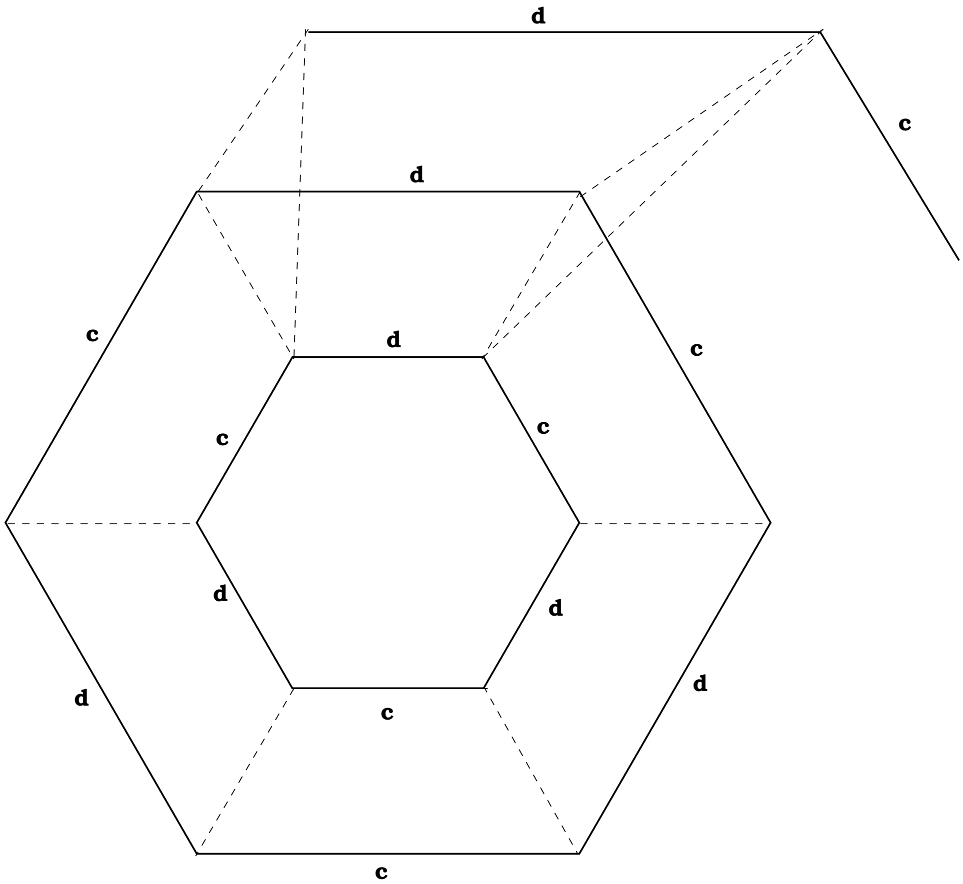

Figure 5. Alternating (c, d)-cycles to illustrate Lemma 5.4.

Proof. Let $d \in S \cap H$ and $c \in S \setminus H$.

It follows from Lemma 5.3, that an integer k such that $d(cd)^k = 1$ does not exist (otherwise elements c and d would commute). We put $m = cdc$. Apparently, m is a member of H and there does not exist l such that $d(md)^l = 1$. Therefore, we can use Lemma 5.4 to find a set T of vertex disjoint spanning cycles of $\mathrm{Cay}(H, S_1)$, where $S_1 = \{d, d^{-1}, m, m^{-1}\}$.

Now, let us take any cycle $v_1, v_2, ..., v_{2k}$ from S_1. We put the flow f on this cycle's edges equal one, $f(v_i, v_{i+1}) = 1$ for $i \in \langle 1, 2k-1 \rangle$. The flow f on the edges not contained in S_1 will be 0.

At last, we replace any (g, dg)-edges in flow by path $g, ag, acg, acag$. The function f is now 3-flow on c-edges and a-edges such that the flow value of any oriented c-edge equals 1 or 0. Any a-edge has nonzero flow value. Therefore, the flow f_2 defined on a-edges and c-edges as

- $f_2(u, cu) = f_1(u, cu) + 1$
- $f_2(u, au) = f_1(u, au)$

is a nowhere-zero 3-flow on $\mathrm{Cay}(G, S)$. Since the graph $\mathrm{Cay}(G, S)$ admits a nowhere-zero 3-flow, it is bipartite. □

Theorem 5.2 *Let G be a group containing an Abelian subgroup of index two. Then every Cayley graph $\mathrm{Cay}(G, S)$ of valency at least four has a nowhere-zero 3-flow.*

Proof. We prove lemma by induction on $|G|$. If $|G| = 2$, then there is nothing to show. The graph is bipartite and has a nowhere-zero 3-flow. So we can use it as the induction start.

For the induction step, let us consider four different cases.

CASE 1: $|H \cap S| \leq 2$. Since the index of H in G equals to 2, graph $\mathrm{Cay}(G, S \setminus H)$ is bipartite. Clearly, it has at least three disjoint 1-factors. Therefore, the union of these 1-factors is a cubic bipartite graph and as such has a nowhere-zero 3-flow.

CASE 2: $|H \cap S| = 3$. Clearly, there is at least one involution in the group H. We know from Lemma 4.2, that there is an involution h in H such that $H_1 = \langle h \rangle$ is characteristic subgroup of G. If h belongs to S, then $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow from Theorem 3.1. It follows from Lemma 5.2, that we can apply induction hypothesis to $\mathrm{Cay}(G/H_1, S/H_1)$ and we are done.

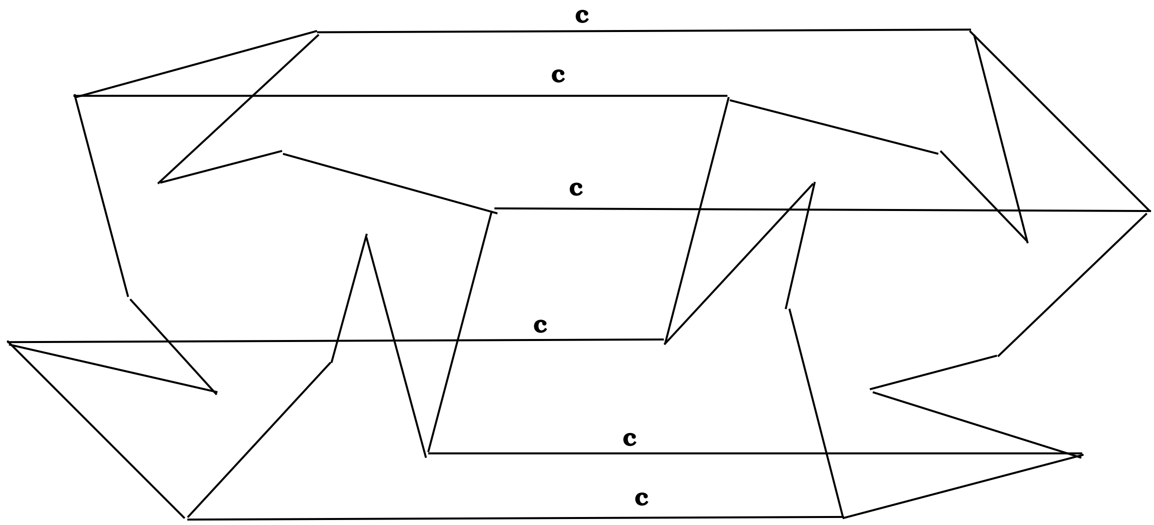

Figure 6. Original Cay$(G, \{c, d\})$ where c is the involution, d is a member normal subgroup H of G of index two. Edges generated by the involution c are horizontal. All other edges are d-edges. The vetices of H are on the letf and vertices of another coset are on the right.

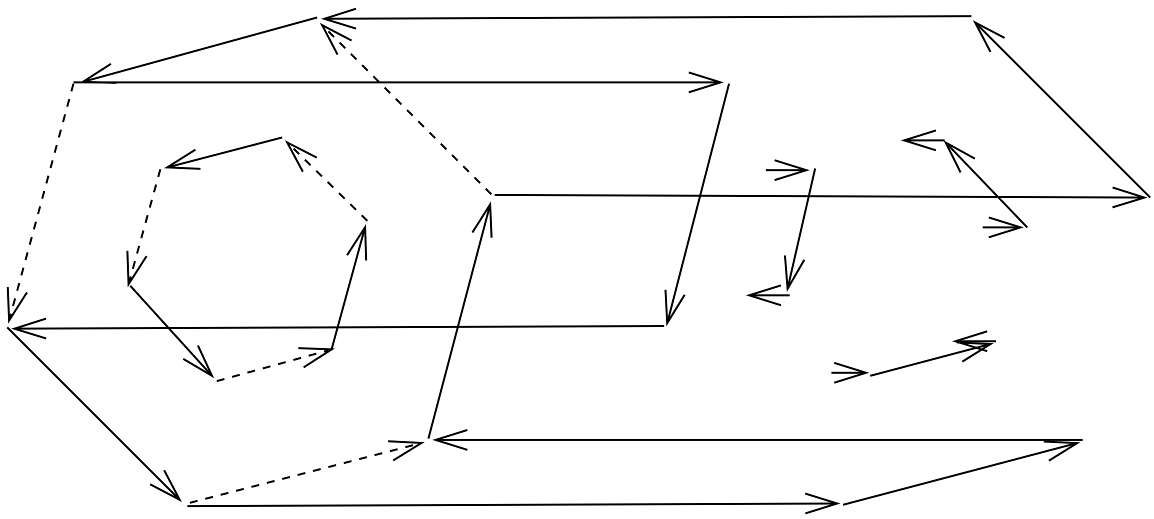

Figure 7. Alternating (m, d)-cycles. Lines generated by $m = cdc$ are dotted. The flow is shown by arrows.

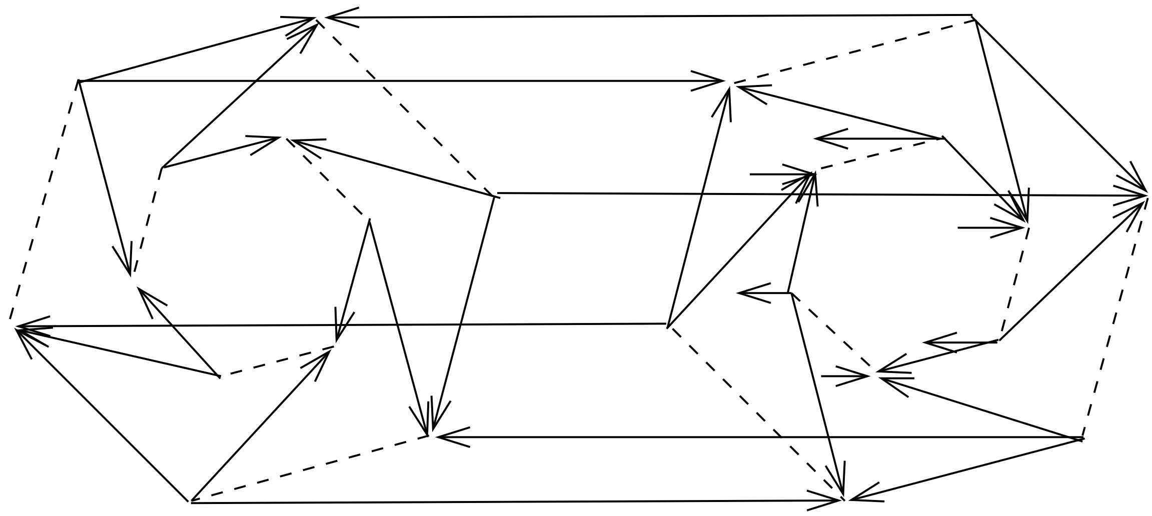

Figure 8. Final nowhere-zero $\mathbb{Z}_3$-flow. Arrows direction represents the flow value.

Case 3: $|H \cap S| = 4$. Now, let a be an element from $S \setminus H$. Thus, a is an involution. We distinguish three different subcases. First of all, if $H \cap S$ consists of four involutions, then there is a cubic bipartite spanning Cayley subgraph. Therefore $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow.

For the second subcase, let $H \cap S$ consist of exactly two involutions and one non-involution. Again, there is h in H such that $H_1 = \{e, h\}$ is characteristic subgroup of G. Proof is then analogous to the case $|H \cap S| = 3$, the existence of a nowhere-zero 3-flow is a consequence of Lemma 5.2 and Lemma 4.1.

Finally, let $H \cap S = \{c, c^{-1}, d, d^{-1}\}$. If either c or d do not commute with a, then existence of a nowhere-zero 3-flow is straigthforward consequence of Theorem 5.1 We may therefore assume that a $ca = ac$ and $da = ad$. Then, a is a central involution in $\langle a, c, d\rangle$. Therefore, $\mathrm{Cay}(G, S)$ has a nowhere-zero 3-flow.

Case 4: $|H \cap S| = 5$. $\mathrm{Cay}(G, S)$ consist of 2 Cayley graphs on an Abelian group and as such has a nowhere-zero 3-flow. □

As a consequence of the last theorem, all Cayley graphs $\mathrm{Cay}(G, S)$ of valency at least four where G is solvable containing an Abelian normal subgroup of index two admit a nowhere-zero 3-flow.

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