Comenius University, Bratislava
Faculty of Mathematics, Physics and Informatics

Higher-Order Description Logics for Metamodelling

Master's Thesis

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Comenius University, Bratislava
Faculty of Mathematics, Physics and Informatics

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Study programme:	Informatics
Study field:	2508 Informatics
Department:	Department of Computer Science
Supervisor:	Mgr. Ján Kľuka, PhD.
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THESIS ASSIGNMENT

Name and Surname:	Bc. Petra Kubincová
Study programme:	Computer Science (Single degree study, master II. deg., full time form)
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Title: Higher-Order Description Logics for Metamodelling

Aim: Investigate the extension of higher-order description logics proposed by Homola et al. [1] with classes of non-homogeneous instances and roles with non-homogeneous domains and ranges. Additionally, investigate any of the following extensions: classes of roles; classes and roles with set-theoretic semantics; existential and universal restrictions over the instance-of and subclass-of relations.
Propose suitable syntax and semantics for the extensions. Investigate decidability and complexity of standard reasoning tasks.

Literature:
[1] Homola, M., Kľuka, J., Svátek, V., Vacura, M. Typed Higher-Order Variant of SROIQ – Why Not? In: Bienvenu, M., et al. (eds.): DL 2014 – 27th International Workshop on Description Logics. CEUR Workshop Proceedings 1193, ceur-ws.org (2014).
[2] Svátek, V., Homola, M., Kľuka, J., Vacura, M.: Metamodeling-based coherence checking of OWL vocabulary background models. In: OWLED (2013).
[3] Motik, B.: On the properties of metamodeling in OWL. J. Log. Comput 17(4), 617–637 (2007).
[4] Glimm, B., Rudolph, S., Völker, J.: Integrated metamodeling and diagnosis in OWL 2. In: ISWC (2010).
[5] De Giacomo, G., Lenzerini, M., Rosati, R.: On higher-order description logics. In: DL (2009).
[6] Pan, J.Z., Horrocks, I.: RDFS(FA): Connecting RDF(S) and OWL DL. IEEE Trans. Knowl. Data Eng. 19, 192–206 (2007).

Annotation: Homola et al. [1] have proposed a practically-motivated [2] typed higher-order version of a family of description logics up to SROIQ. This work is related to earlier efforts by Motik [3], Glimm et al. [4], De Giacomo et al. [5], Pan and Horrocks [6], and others. The logics of Homola et al. allow for modelling certain higher-order entities (homogeneous concepts of concepts, roles between concepts of any order with homogeneous domains and ranges), and have (unlike set theory) a non-extensional semantics suitable for domain meta modelling use cases.
There are numerous open problems in these logics, some of which have been selected for this thesis assignment.

Keywords: description logics, higher-order logic, theory of types, meta modelling

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ZADANIE ZÁVEREČNEJ PRÁCE

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Sekundárny jazyk: slovenský

Názov: Higher-Order Description Logics for Metamodelling
Deskripčné logiky vyšších rádov pre metamodelovanie

Cieľ: Preskúmať rozšírenie deskripčných logík vyšších rádov navrhnutých Homolom a kol. [1] o triedy nehomogénnych inštancií a roly s nehomogénnymi obormi definície a hodnôt. Ďalej preskúmať niektoré z nasledujúcich rozšírení: triedy rol, triedy a roly s množinovoteoretickou sémantikou, existenčné a univerzálne reštrikcie nad reláciami byť inštanciou a byť podtriedou.
Pre rozšírenia je potrebné navrhnúť vhodnú syntax a sémantiku, preskúmať rozhodnuteľnosť a zložitosť štandardných problémov vyplývania.

Literatúra:

[1] Homola, M., Kľuka, J., Svátek, V., Vacura, M. Typed Higher-Order Variant of SROIQ – Why Not? In: Bienvenu, M., et al. (eds.): DL 2014 – 27th International Workshop on Description Logics. CEUR Workshop Proceedings 1193, ceur-ws.org (2014).
[2] Svátek, V., Homola, M., Kľuka, J., Vacura, M.: Metamodeling-based coherence checking of OWL vocabulary background models. In: OWLED (2013).
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Anotácia: Homola a kol. [1] navrhli na základe praktickej motivácie [2] verziu rodiny deskripčných logík až po logiku SROIQ s typovanými konštrukciami vyšších rádov. Výsledky vychádzajú z predchádzajúcej práce Motika [3], Glimmovej a kol. [4], De Giacoma a kol. [5], Pana a Horrocksa [6] a ďalších. Logiky podľa Homolu a kol. umožňujú modelovanie niektorých entít vyšších rádov (homogénne koncepty konceptov, roly medzi konceptmi rôznych rádov, pokiaľ sú obory definície a hodnôt homogénne) a majú (na rozdiel od teórie množín) neextenzionálnu sémantiku vhodnú pre účely meta modelovania.
Vyriešenie niektorých z množstva otvorených problémov v uvedených logikách sme vybrali ako ciele tohto zadania diplomovej práce.

Kľúčové slová: deskripčné logiky, logiky vyšších rádov, teória typov, meta modelovanie

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študent vedúci práce

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Abstrakt

Existuje mnoho deskripčných logík líšiacich sa expresivitou, pričom všetky z nich sú fragmentami prvorádovej predikátovej logiky. Preto v nich nie je možné priamočiaro modelovať domény, ktorých štruktúra prirodzene obsahuje vyššie rády. Štandardným deskripčným logikám tiež chýbajú prostriedky, ktoré by umožnili modelovanie s niektorými jazykovými operátormi, napríklad s operátorom inštanciácie, nad rámec ich bežného použitia.

V tejto práci navrhujeme štyri deskripčné logiky vyšších rádov s metamodelovacími prvkami, ktoré dovoľujú voľne modelovať s inštanciáciou a čiastočne tiež so subsumpciou. Navyše ukážeme, že naše logiky majú aj iné vlastnosti vhodné pre metamodelovanie. Dokážeme rozhodnuteľnosť našich logík redukciou na štandardné deskripčné logiky. Ďalej porovnávame naše logiky s už existujúcimi logikami vyšších rádov.

Keďže redukcia, ktorú používame, je polynomiálna, s našimi logikami môžu pracovať algoritmy určené pre štandardné deskripčné logiky, pričom zložitosť bude rovnaká ako pre štandardné logiky. Naša redukcia navyše dokazuje, že expresívna sila potrebná na istý typ modelovania s vyššími rádmi je prítomná už aj v štandardných deskripčných logikách.

Kľúčové slová: deskripčné logiky, logika vyšších rádov, teória typov, metamodelovanie

Abstract

While there exist many description logics (DLs) with various expressivity, all of them are fragments of first-order logic. Thus, some domains with inherent higher-order structure cannot be straightforwardly modelled even in the most expressive standard DLs. Such standard DLs also lack the features that would allow to freely model with some of the language operators, e.g., the instantiation operator.

In this thesis, we propose four higher-order description logics with metamodelling features allowing to freely model with instantiation and partially also with subsumption. In addition, we show that our higher-order DLs have also other properties desirable for metamodelling. We prove their decidability by means of reduction to standard DLs. Further, we compare our higher-order DLs with other existing higher-order DLs.

Since the reduction is polynomial, our higher-order DLs can be decided by algorithms for standard DLs with the same complexity. Moreover, the reduction shows that the expressive power needed to model with higher orders to some extent is already present in the standard description logics.

Keywords: description logics, higher-order logic, theory of types, metamodelling

Contents

Introduction

Description logics are (usually) decidable fragments of first-order logic. They are used for knowledge representation, namely for modelling ontologies, i.e., systems of concepts and relationships in domains. Besides, they constitute a formal basis of Semantic Web language OWL. Various description logics have been built over time from the less expressive description logics, as $\mathcal{EL}$ and $\mathcal{ALC}$, up to $\mathcal{SROIQ}$, a very expressive yet decidable standard description logic.

While many domains can be modelled with description logics, there are certain situations when the expressivity of standard description logics is not sufficient. One of these situations is modelling domains in which higher orders arise naturally. Consider the biological taxonomy, where individual animals are classified into various taxa (e.g., Melman is an instance of the species Giraffa camelopardalis), while the taxa themselves are classified into ranks (e.g., Giraffa camelopardalis is an instance of the rank species). Since description logics are fragments of the first-order logic, they cannot fully capture the relationships between higher-order concepts. For example, in standard description logics concepts cannot be instances of other concepts and thus Giraffa camelopardalis cannot be an instance of species.

Although some works already tackled this problem (Pan et al., 2005; Motik, 2007; Cuenca Grau et al., 2008; Glimm et al., 2010; De Giacomo et al., 2011; Homola et al., 2014; Motz et al., 2015), none of them had all the properties and features we consider desirable for metamodelling. Some approaches lacked unlimited higher orders, allowing only second-order concepts and most approaches lacked the possibility to model freely with selected language operators, such as instantiation or subsumption. While some approaches allowed only strictly typed concepts (i.e., concepts separated into layers), some approaches did not even allow typing of selected concepts. Hence, we decided to extend $\mathcal{SROIQ}$ with expressive higher-order and metamodelling features fulfilling our requirements for a higher-order description logic, while retaining decidability.

Our contribution consists of four higher-order extensions of description logics with different properties. All of them feature unlimited higher orders, optional typing of concepts and the possibility to freely model with the instantiation relationship. They also have several properties desirable for metamodelling (see Chapter 5). Three of our extensions also feature different ways of modelling with the subsumption relationship. Further, all of these extensions are polynomially reducible to the description logic they extend (under some technical assumptions), which allows to decide them with the same computational complexity as the underlying description logic.

Parts of this work were already published in the following papers:

- Petra Kubincová, Ján Kľuka, and Martin Homola (2015). *Towards Expressive Metamodelling with Instantiation.* Presented at the 28th International Workshop on Description Logics in Athens, Greece.
- Petra Kubincová, Ján Kľuka, and Martin Homola (2016). *Expressive Description Logic with Instantiation Metamodelling.* Accepted for a presentation at the 15th International Conference on Principles of Knowledge Representation and Reasoning in Cape Town, South Africa. To appear in proceedings published by AAAI Press.

This thesis starts with introduction of the description logics area describing $\mathcal{SROIQ}$ and three other description logics (Chapter 1). Then we specify the domain featuring higher-orders which motivated our work (Chapter 2) and we explore selected related work (Chapter 3). Finally, we introduce and study our higher-order description logics extensions (Chapter 4). Last but not least, in Chapter 5 we discuss the properties of our contribution and compare it to the related work from Chapter 3. We conclude the thesis with a summary and an outline of future work.

Chapter 1

Introduction to Description Logics

Description logics (DLs) are a family of logics used primarily in knowledge representation, e.g., to build *ontologies*. Ontology is a formal conceptualization (description) of some domain. An example of a domain, which we use further in this chapter, is a zoo with animals (e.g., giraffe Melman) and their caretakers (e.g., human John). In the ontology, we can classify individuals (e.g., Melman is a herbivore), model relationships between individuals (e.g., John feeds Melman) and even some relationships between classes of individuals (e.g., all girrafes are herbivores). As we will see later, DLs are capable of expressing also much more complicated assertions.

DLs can be seen as fragments of the first-order (predicate) logic (FOL). As opposed to FOL, logical inference in DLs is often decidable, and such logics are themselves called decidable. DLs differ in expressivity and, consequently, in complexity of decidability.

In this chapter we introduce the basis of our work, the DL $\mathcal{SROIQ}$, and DLs $\mathcal{SHOIQ}$, $\mathcal{SHIQ}$ and $\mathcal{ALCHOIQ}$. All these logics are results of an effort to create expressive yet decidable DL. $\mathcal{SROIQ}$ is the most expressive one – $\mathcal{SHOIQ}$, $\mathcal{SHIQ}$ and $\mathcal{ALCHOIQ}$ were created before $\mathcal{SROIQ}$ and thus are less expressive and can be viewed as fragments of $\mathcal{SROIQ}$.

1.1 $\mathcal{SROIQ}$

The description logic $\mathcal{SROIQ}$ (Horrocks et al., 2006) is considered a generally accepted standard for a very expressive DL. It is used as a basis for Web ontology language 2 (OWL 2, Cuenca Grau et al. (2008)).

In this section we introduce the syntax and semantics of $\mathcal{SROIQ}$, we show some examples of use and we discuss its decidability.

1.1.1 Syntax

We start with defining the $\mathcal{SROIQ}$ vocabulary.

Definition 1.1 ($\mathcal{SROIQ}$ vocabulary)**.** *A $\mathcal{SROIQ}$* vocabulary *is a triple of mutually disjoint countable sets of names* $(N_{\mathrm{I}}, N_{\mathrm{C}}, N_{\mathrm{R}})$ *– the set of* individual names N_{I}*, the set of* concept names N_{C} *and the set of* role names N_{R} *– where* universal role $\mathsf{U} \in N_{\mathrm{R}}$.

All standard description logics (i.e., well-known fragments of $\mathcal{SROIQ}$) have very similar definitions of the vocabulary (most of the fragments do not require $\mathsf{U} \in N_{\mathrm{R}}$).

Individual names represent particular objects, concept names represent classes of individuals and roles represent relationships between pairs of individuals. (Individual names correspond to FOL constants, concept names to FOL unary predicates and role names to FOL binary predicates.)

Let us introduce a vocabulary for an example modelling classification and a relationships of zoo animals and their caretakers mentioned in the introduction to this chapter. Our example involves individuals melman (a giraffe), john (a human), and zooNewYork (a zoo). Individuals can be classified in concepts Giraffa camelopardalis (the only extant giraffe species), Giraffa (the genus classifying giraffes), Homo sapiens (the only extant human species), Zoo (classifying zoos) and Herbivore (classifying herbivores). Relationships will be modelled by roles livesIn, livedIn and feeds. For example, melman is an animal of species Giraffa camelopardalis whose relationship to zooNewYork can be represented by role livesIn. The convention is to start the names of concepts with a capital letter while the names of individuals and roles are lower camel case.

$$\begin{aligned} N_{\mathrm{I}} &= \{\mathsf{melman}, \mathsf{john}, \mathsf{zooNewYork}\} \\ N_{\mathrm{C}} &= \{\mathsf{Giraffa\ camelopardalis}, \mathsf{Giraffa}, \mathsf{Homo\ sapiens}, \mathsf{Zoo}, \mathsf{Herbivore}\} \\ N_{\mathrm{R}} &= \{\mathsf{livesIn}, \mathsf{livedIn}, \mathsf{feeds}\} \end{aligned} \tag{1.1}$$

Next, we define $\mathcal{SROIQ}$ roles and role chains, i.e., complex role expressions, which can then be used to build some assertions or more complex descriptions of classes.

Definition 1.2 ($\mathcal{SROIQ}$ roles and role chains)**.** *For each $R \in N_{\mathrm{R}}$, both R and R^{-} (inverse of R) are $\mathcal{SROIQ}$* roles*. We will denote the set of all roles with respect to some N_{R} by $\mathbf{R}$.*

For $R_1, \ldots, R_n \in \mathbf{R}$, $R_1 \cdots R_n$ is a role chain*.*

The inverse of a role is intended to denote a role where the order of connected individuals is reversed, just like the inverse of a binary relation. E.g., livesIn connects individual melman with individual zooNewYork, while $\mathsf{livesIn}^{-}$ connects zooNewYork with melman. Role chains "join" more roles into one. E.g., if a zookeeper called john feeds melman, then $\mathsf{feeds} \cdot \mathsf{livesIn}$ connects john with zooNewYork.

With roles we can already model a simple ontology. This is done by describing the domain with *axioms* – formal sentences consisting of elements of the $\mathcal{SROIQ}$ language.

Definition 1.3 ($\mathcal{SROIQ}$ axioms I.)**.** *Let w be a role chain and R, P (inverse) roles. Then $w \sqsubseteq R$ is a* role inclusion axiom *(RIA) and* $\mathrm{Ref}(R)$*,* $\mathrm{Irr}(R)$*,* $\mathrm{Sym}(R)$*,* $\mathrm{Tra}(R)$*,* $\mathrm{Dis}(R, P)$ *are* role assertions*. A finite set of RIAs is called a* role hierarchy*.*

Now we can state some basic facts using roles from the example vocabulary (1.1). E.g., $\mathrm{Irr}(\mathsf{livesIn})$ expresses that the relationship of living somewhere is irreflexive and $\mathsf{livesIn} \sqsubseteq \mathsf{livedIn}$ states that if someone live*s* somewhere, they also live*d* there. Generally, all RIAs $w \sqsubseteq R$ express that if two individuals are connected through the role chain w, they are connected through the role R, i.e., R is a composition of the roles w. Similarly to the symmetry, $\mathrm{Ref}(R)$ states reflexivity, $\mathrm{Sym}(R)$ symmetry, $\mathrm{Tra}(R)$ transitivity and $\mathrm{Dis}(R, P)$ role disjointness.

RIAs are perhaps the most expressive element of $\mathcal{SROIQ}$. Through the RIAs, roles can depend on each other. These dependencies are closely related with decidability. It is thus important to define what "good" dependencies look like.

Definition 1.4 (RIA and role hierarchy regularity)**.** *A strict partial order on roles $\prec$ is called* regular *when $S \prec R$ iff $S^- \prec R$.*

Let $\prec$ be a regular order on roles. A RIA $w \sqsubseteq R$ for a role chain w and a role R is $\prec$-regular *iff: $w = RR$ or $w = R^-$ or $w = S_1 \cdots S_n$ or $w = R \cdot S_1 \cdots S_n$ or $w = S_1 \cdots S_n \cdot R$, with $S_i \prec R$ for all $1 \leq i \leq n$.*

A role hierarchy is regular *if there exists a regular order $\prec$ on roles such that each RIA of the hierarchy is $\prec$-regular.*

Similarly to roles, also concepts in $\mathcal{SROIQ}$ can be more complex than a concept name. Complex concepts are inductively build from concept names using various constructors.

Definition 1.5 ($\mathcal{SROIQ}$ concepts)**.** *An expression C is a $\mathcal{SROIQ}$* concept *if it is of one of the following forms: $A \in N_\mathrm{C}$, $\neg D_1$ (complement), $D_1 \sqcap D_2$ (intersection), $D_1 \sqcup D_2$ (union), $\forall R.D_1$ (universal restriction), $\exists R.D_1$ (existential restriction), $\{a\}$ (nominal), $\geqslant nR.D_1$ (at-least/qualified number restriction), $\leqslant nR.D_1$ (at-most/qualified number restriction) and $\exists R.\mathsf{Self}$ (self restriction), where $a \in N_\mathrm{I}$, $n \in \mathbb{N}$, D_1, D_2 are concepts and R is a role.*

Concepts belonging to N_C are called atomic concepts*, non-atomic concepts are called* complex.

We will denote the set of all concepts with respect to some N_C by $\mathbf{C}$*.*

For example, on the intuitive level, the complex concept $\neg\mathsf{Giraffa}$ clasiffies every individual that is not a giraffe. The universal restriction $\forall\mathsf{livedIn}.\mathsf{Zoo}$ classifies every individual that lived only in a zoo, while the existential restriction $\exists\mathsf{livedIn}.\mathsf{Zoo}$ classifies every individual that lived in at least one zoo. The intersection $\mathsf{Giraffa}\sqcap\exists\mathsf{livedIn}.\mathsf{Zoo}$ classifies giraffes which lived in at least one zoo. Similarly, the union $\mathsf{Giraffa}\sqcup\exists\mathsf{livedIn}.\mathsf{Zoo}$ classifies all giraffes and all other individuals that lived in at least one zoo. The nominal $\{\mathsf{melman}\}$ represents a concept classifying exactly one individual: melman. The qualified restriction $\geqslant n\,\mathsf{livedIn}.\mathsf{Zoo}$ ($\leqslant n\,\mathsf{livedIn}.\mathsf{Zoo}$) classifies every individual that lived in at least n (at most n) zoos. The self restriction $\exists\mathsf{feeds}.\mathsf{Self}$ classifies every individual that feeds itself (e.g., not nestlings, which are fed by their parents).

In some cases it is useful to have a concept classifying everything and a concept classifying nothing. For these cases, we introduce symbols $\top$ (classifying every individual) and $\bot$ (classifying nothing) as shorthands for $A\sqcup\neg A$ and $A\sqcap\neg A$, for some $A\in N_{\mathrm{C}}$, respectively. Now we can model, e.g., the concept of inhabited places $\exists\mathsf{livesIn}^{-}.\top$.

Definition 1.6 ($\mathcal{SROIQ}$ axioms II.)**.** *Let $a,b\in N_{\mathrm{I}}$, $C,D\in\mathbf{C}$ and $R\in\mathbf{R}$. Then $C\sqsubseteq D$ is a* general concept inclusion *(GCI) and $a\colon C$, $a,b\colon R$, $a,b\colon\neg R$, $a=b$, $a\neq b$ are* individual assertions*.*

Now we can state more facts about our ontology: e.g., $\mathsf{melman}\neq\mathsf{john}$ (melman is not john), $\mathsf{Giraffa\ camelopardalis}\sqsubseteq\mathsf{Giraffe}$ (individuals of species $\mathsf{Giraffa\ camelopardalis}$ belong to genus $\mathsf{Giraffa}$), $\mathsf{Homo\ sapiens}\sqsubseteq\neg\mathsf{Giraffa}$ (humans are not giraffes), $\mathsf{melman}\colon\mathsf{Giraffa\ camelopardalis}$ (melman is a giraffe) and $\mathsf{melman},\mathsf{zooNewYork}\colon\mathsf{livesIn}$ (melman lives in $\mathsf{zooNewYork}$).

$C\equiv D$ is widely used as an abbreviation for two axioms $C\sqsubseteq D$ and $D\sqsubseteq C$. Generally, $C\sqsubseteq D$ (where $\sqsubseteq$ is called *subsumption*, $C\sqsubseteq D$ is read as "D subsumes C") expresses that D is a more general concept than C. Individual assertions $a\colon C$ state that individual a belongs to concept C, $a,b\colon R$ ($a,b\colon\neg R$) states that individuals a,b are (not) connected by role R and $a=b$ ($a\neq b$) states that individuals a,b are (not) equal.

With $\mathcal{SROIQ}$ axioms (counterparts of FOL formulae) defined, we can define the DL counterpart of FOL theory: knowledge base.

Definition 1.7 ($\mathcal{SROIQ}$ knowledge base)**.** $\mathcal{SROIQ}$ knowledge base *(KB) has three parts:* ABox*,* TBox *and* RBox.

ABox is a finite set of individual assertions.

TBox is a finite set of GCIs.

RBox is a union of a finite set of role assertions with a role hierarchy. RBox is regular if its role hierarchy is regular.

ABox is usually denoted by $\mathcal{A}$, TBox by $\mathcal{T}$, RBox by $\mathcal{R}$, and knowledge base by $\mathcal{K}$.

An example of a knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ using the example vocabulary (1.1) containing some of the already mentioned axioms follows.

$$\begin{aligned}
\mathcal{A} &= \{\mathsf{melman}\colon \mathsf{Giraffa\ camelopardalis}, \\
&\quad\ \mathsf{zooNewYork}\colon \mathsf{Zoo}, \\
&\quad\ \mathsf{melman}, \mathsf{zooNewYork}\colon \mathsf{livesIn}, \\
&\quad\ \mathsf{john}, \mathsf{melman}\colon \mathsf{feeds}\} \\
\mathcal{T} &= \{\mathsf{Giraffa\ camelopardalis} \sqsubseteq \mathsf{Giraffa}, \\
&\quad\ \mathsf{Giraffa} \sqsubseteq \mathsf{Herbivore}, \\
&\quad\ \mathsf{Herbivore} \sqcap \mathsf{Zoo} \sqsubseteq \bot, \\
&\quad\ \exists \mathsf{livesIn}.\mathsf{Zoo} \sqsubseteq \forall \mathsf{feeds}^{-}.\mathsf{Homo\ sapiens}\} \\
\mathcal{R} &= \{\mathsf{livesIn} \sqsubseteq \mathsf{livedIn}\}
\end{aligned} \tag{1.2}$$

The example ABox states that melman is a giraffe, zooNewYork is a zoo, melman lives in zooNewYork and john feeds melman. The TBox expresses that every member of species Giraffa camelopardalis is also a member of genus Giraffa, that giraffes are herbivores, herbivores and zoos are disjoint and that everything that lives in some zoo is fed only by humans. The RBox states that to have lived somewhere is more general than to live somewhere.

This example is by no means exhaustive – there could be many more axioms added to complete the picture, e.g., $\mathsf{john}\colon \mathsf{Homo\ sapiens}$ or $\mathsf{Giraffa} \sqcap \mathsf{Homo\ sapiens} \sqsubseteq \bot$.

1.1.2 Semantics

$\mathcal{SROIQ}$ semantics is first-order. Individuals are interpreted as elements of the domain set, concepts as subsets of the domain set and roles as relations on the domain set.

Definition 1.8 ($\mathcal{SROIQ}$ interpretation)**.** *An* interpretation *of a* $\mathcal{SROIQ}$ *KB is a pair* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *where* $\Delta^{\mathcal{I}} \neq \emptyset$ *is called* domain *and* $\cdot^{\mathcal{I}}$ *is an* interpretation function *satisfying the following conditions:*

1. $\forall a \in N_{\mathrm{I}} : a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
2. $\forall A \in N_{\mathrm{C}} : A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
3. $\forall R \in N_{\mathrm{R}} : R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

The interpretation function is inductively extended to $\mathbf{C}$*,* $\mathbf{R}$ *and role chains according to Table 1.1 (for* $a \in N_{\mathrm{I}}$*,* $n \in \mathbb{N}$*,* $C, D \in \mathbf{C}$*,* $R_0 \in N_{\mathrm{R}}, R \in \mathbf{R}$ *and a role chain* $R_1 \cdots R_n$*).*

Table 1.1: Syntax and Semantics of $\mathcal{SROIQ}$ Expressions

Syntax (x)	Semantics ($x^{\mathcal{I}}$)
R_0	$R_0^{\mathcal{I}}$
R^-	$\{ (y,x) \mid (x,y) \in R^{\mathcal{I}} \}$
U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
$R_1 \cdots R_n$	$R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}}$
A	$A^{\mathcal{I}}$
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\forall R.C$	$\{ x \mid \forall y.(x,y) \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}} \}$
$\exists R.C$	$\{ x \mid \exists y.(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}} \}$
$\{a\}$	$\{ a^{\mathcal{I}} \}$
$\geqslant nR.C$	$\{ x \mid \#\{ y \mid (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}} \} \geq n \}$
$\leqslant nR.C$	$\{ x \mid \#\{ y \mid (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}} \} \leq n \}$
$\exists R.\mathsf{Self}$	$\{ x \mid (x,x) \in R^{\mathcal{I}} \}$

For an example, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be defined as in the following example.

$$\begin{aligned} \Delta^{\mathcal{I}} &= \{m, z\} \\ \mathsf{melman}^{\mathcal{I}} &= \mathsf{john}^{\mathcal{I}} = m \\ \mathsf{zooNewYork}^{\mathcal{I}} &= z \\ \mathsf{Giraffa\ camelopardalis}^{\mathcal{I}} &= \mathsf{Giraffa}^{\mathcal{I}} = \mathsf{Homo\ sapiens}^{\mathcal{I}} = \mathsf{Herbivore}^{\mathcal{I}} = \{m\} \\ \mathsf{Zoo}^{\mathcal{I}} &= \{z\} \\ \mathsf{livesIn}^{\mathcal{I}} &= \mathsf{livedIn}^{\mathcal{I}} = \{(m,z)\} \\ \mathsf{feeds}^{\mathcal{I}} &= \{(m,m)\} \end{aligned} \tag{1.3}$$

Then $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is an interpretation of the example knowledge base (1.2). Note that while $\mathcal{I}$ is an interpretation of this KB, it is not the intuitive interpretation one instantly thinks of when looking at the KB, since it interprets melman and john as the same element of the domain set.

Note that the above definition makes some of the constructors redundant. E.g., the semantics of $C \sqcup D$ is the same as the semantics of $\neg(\neg C \sqcap \neg D)$. In the following we will sometimes omit constructors which meaning can be expressed by other constructors: union ($\sqcup$), universal restriction ($\forall R.C$, as its meaning can be expressed as $\neg\exists R.\neg C$) and at-most qualified restriction ($\leqslant nR.C$, as its meaning can be expressed as $\neg\geqslant(n+1)R.C$).

Now we can define satisfiability of axioms – i.e., the conditions under which the

Table 1.2: Syntax and Semantics of $\mathcal{SROIQ}$ Axioms

Axiom φ	Semantic constraint
$C \sqsubseteq D$	$C^{\mathcal{E}} \subseteq D^{\mathcal{I}}$
$a\colon C$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
$a, b\colon R$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
$a, b\colon \neg R$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$
$a = b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$
$w \sqsubseteq R$	$w^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\mathrm{Ref}(R)$	$\forall x \in \Delta^{\mathcal{I}}.(x, x) \in R^{\mathcal{I}}$
$\mathrm{Irr}(R)$	$\forall x \in \Delta^{\mathcal{I}}.(x, x) \notin R^{\mathcal{I}}$
$\mathrm{Sym}(R)$	$\forall x, y.(x, y) \in R^{\mathcal{I}} \Rightarrow (y, x) \in R^{\mathcal{I}}$
$\mathrm{Tra}(R)$	$\forall x, y, z.(x, y) \in R^{\mathcal{I}} \wedge (y, z) \in R^{\mathcal{I}} \Rightarrow (x, z) \in R^{\mathcal{I}}$
$\mathrm{Dis}(R, P)$	$R^{\mathcal{I}} \cap P^{\mathcal{I}} = \emptyset$

axioms hold.

Definition 1.9 ($\mathcal{SROIQ}$ satisfiability)**.** *An axiom φ is* satisfied *by a $\mathcal{SROIQ}$ interpretation $\mathcal{I}$ ($\mathcal{I} \models \varphi$) if $\mathcal{I}$ satisfies the respective semantic constraint from Table 1.2 (for $a, b \in N_{\mathrm{I}}$, $n \in \mathbb{N}$, $C, D \in \mathbf{C}$, $R, P \in \mathbf{R}$ and a role chain w).*

A $\mathcal{SROIQ}$ interpretation $\mathcal{I}$ is a model *of $\mathcal{K}$ ($\mathcal{I} \models \mathcal{K}$) if $\mathcal{I}$ satisfies every axiom $\varphi \in \mathcal{K}$.*

A concept C is satisfiable in *$\mathcal{K}$ if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.*

An axiom φ is entailed *by $\mathcal{K}$ ($\mathcal{K} \models \varphi$) if $\mathcal{I} \models \varphi$ holds for each $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$.*

For example, $\exists \mathsf{livesIn}.\mathsf{Zoo} \sqsubseteq \forall \mathsf{feeds}^{-}.\mathsf{Homo\ sapiens}$ is satisfied by the example interpretation (1.3), because $(\exists \mathsf{livesIn}.\mathsf{Zoo})^{\mathcal{I}} = \{m\}$, $(\forall \mathsf{feeds}^{-}.\mathsf{Homo\ sapiens})^{\mathcal{I}} = \{m\}$ and $\{m\} \subseteq \{m\}$. Likewise, every axiom from the example KB $\mathcal{K}$ (1.2) is satisfied by the example interpretation $\mathcal{I}$ (1.3). Thus, $\mathcal{I}$ from (1.3) is a model of the KB $\mathcal{K}$ from (1.2): $\mathcal{I} \models \mathcal{K}$ (though it certainly is not the intended, intuitive model). Although the intepretation $\mathcal{I}$ (1.3) is a model of KB $\mathcal{K}$ (1.2), it does not, of course, satisfy all axioms in its vocabulary (1.1), e.g., $\mathsf{john} : \neg\mathsf{Herbivore}$, because it is not true that $\mathsf{john}^{\mathcal{I}} = m \in \Delta^{\mathcal{I}} \setminus \mathsf{Herbivore}^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus \{m\} = \{z\}$, or $\mathsf{Giraffa} \sqcap \mathsf{Homo\ sapiens} \sqsubseteq \bot$, because it is not true that $\mathsf{Giraffa}^{\mathcal{I}} \cap \mathsf{Homo\ sapiens}^{\mathcal{I}} = \{m\} \cap \{m\} \subseteq \emptyset = \bot^{\mathcal{I}}$.

Note again that the above definition makes some axioms redundant. For example, the condition under which $\mathrm{Tra}(R)$ holds is the same as for $R \cdot R \sqsubseteq R$. In the following we will sometimes omit axiom forms which meaning can be expressed by other axiom forms: $\mathrm{Tra}(R)$, $\mathrm{Ref}(R)$ (as its meaning can be expressed as $\top \sqsubseteq \exists R.\mathsf{Self}$), $\mathrm{Irr}(R)$

(expressible by $\top \sqsubseteq \neg\exists R.\mathsf{Self}$), $\mathrm{Sym}(R)$ (expressible by $R^- \sqsubseteq R$), $a = b$ (expressible by $\{a\} \equiv \{b\}$) and $a \neq b$ (expressible by $\{a\} \sqcap \{b\} \sqsubseteq \bot$).

To find out whether some axiom is entailed by a knowledge base is usually not as simple as checking whether it is satisfied by some interpretation. E.g., is $\mathsf{john}\colon \mathsf{Homo\ sapiens}$ entailed by the example knowledge base (1.2)? Let's see what must hold for an interpretation satisfying the knowledge base: Let $j = \mathsf{john}^{\mathcal{I}}$, $m = \mathsf{melman}^{\mathcal{I}}$ and $z = \mathsf{zooNewYork}^{\mathcal{I}}$. Because of the axiom $\mathsf{john}, \mathsf{melman}\colon \mathsf{feeds}$, surely $(j, m) \in \mathsf{feeds}^{\mathcal{I}}$. Moreover, since $\mathsf{melman}, \mathsf{zooNewYork}\colon \mathsf{livesIn}$ and $\mathsf{zooNewYork}\colon \mathsf{Zoo}$, it is true that $m \in (\exists \mathsf{livesIn}.\mathsf{Zoo})^{\mathcal{I}}$. From the axiom $\exists \mathsf{livesIn}.\mathsf{Zoo} \sqsubseteq \forall \mathsf{feeds}^-.\mathsf{Homo\ sapiens}$ follows that $m \in (\forall \mathsf{feeds}^-.\mathsf{Homo\ sapiens})^{\mathcal{I}}$, meaning that $\forall x.(x, m) \in \mathsf{feeds}^{\mathcal{I}} \Rightarrow x \in \mathsf{Homo\ sapiens}^{\mathcal{I}}$. Thus, $j \in \mathsf{Homo\ sapiens}^{\mathcal{I}}$ – the axiom $\mathsf{john}\colon \mathsf{Homo\ sapiens}$ is entailed by the example knowledge base. Problems like this one, i.e., deciding the entailment, are a type of decision problems studied in the description logics area.

1.1.3 Decision Problems and Complexity

For description logics, there are several decision problems which are considered to be standard. First, we define three standard decision problems for DLs and then we show how some of them can be reduced to other ones. Note that these problems are the same for all description logics, but the reductions are not necessarily possible in DLs other than $\mathcal{SROIQ}$.

Definition 1.10 (Decision problems for DLs)**.** *For knowledge base $\mathcal{K}$, $C \in \mathbf{C}$ and an axiom φ, basic decision problems are defined as follows:*

Knowledge base satisfiability *(*consistency*) – $\mathcal{K}$ is* satisfiable *(*consistent*) if it has a model.*

Concept satisfiability *– C is* satisfiable with respect to $\mathcal{K}$ *if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.*

Axiom entailment *– $\mathcal{K}$* entails φ *($\mathcal{K} \models \varphi$) if each model of $\mathcal{K}$ is also a model of φ.*

In the description logics area, "DL $\mathcal{L}$ is decidable with complexity $f(n)$" is often used instead of "axiom entailement in DL $\mathcal{L}$ is decidable with complexity $f(n)$".

For example, example knowledge base $\mathcal{K}$ (1.2) is satisfiable, because it has a model – the example interpretation (1.3). (To model the target domain correctly, KB has to be consistent.) Concept $\top$ is always satisfiable and concept $\bot$ is always unsatisfiable. Since the purpose of concepts is to classify domain objects, a unsatisfiable concept usually indicates an error. Knowledge base containing axioms $\mathsf{melman}\colon \mathsf{Giraffa\ camelopardalis}$ and $\mathsf{Giraffa\ camelopardalis} \sqsubseteq \mathsf{Giraffa}$ entails $\mathsf{melman}\colon \mathsf{Giraffa}$.

In $\mathcal{SROIQ}$, both axiom entailment and concept satisfiability are reducible to knowledge base satisfiblity, and vice versa.

The idea of the reduction for axiom entailment is that for each axiom φ there exists its "negation", $\text{neg}(\varphi)$ – a set of axioms (possibly containing only one axiom) such that φ is satisfied iff $\text{neg}(\varphi)$ is not satisfied. (The definition of $\text{neg}(\varphi)$ for each φ can be found in the work *Foundations of Description Logics* by Rudolph (2011, p. 51).) Then, φ is entailed by $\mathcal{K}$ iff each model of $\mathcal{K}$ is also a model of φ iff there is not a model of $\mathcal{K}$ satisfying $\text{neg}(\varphi)$ iff $\mathcal{K} \cup \text{neg}(\varphi)$ is not satisfiable. Thus, $\mathcal{K} \models \varphi$ iff $\mathcal{K} \cup \text{neg}(\varphi)$ is not satisfiable. For the converse reduction, $\mathcal{K}$ is satisfiable iff $\mathcal{K}$ does not entail a unsatisfiable axiom, e.g., $\top \sqsubseteq \bot$.

For concept satisfiability, the construction is even simpler. C is satisfiable w.r.t. $\mathcal{K}$ iff there exists a model $\mathcal{I}$ of $\mathcal{K}$ s.t. $C^{\mathcal{I}} \neq \emptyset$ iff $\mathcal{K} \cup \{\mathsf{a} \colon C\}$ is satisfiable for a new individual name a. For the converse reduction, $\mathcal{K}$ is satisfiable iff a new concept name C is satisfiable w.r.t. $\mathcal{K}$.

Since there are reductions from axiom entailment to KB satisfiability and from KB satisfiability to concept satisfiability, axiom entailment can be reduced to concept satisfiability and vice versa.

Now we miss only one notion important for decidability: simple roles. In a nutshell, using non-simple roles in some constructions leads to undecidability (Horrocks et al., 1999).

Definition 1.11 (Simple roles)**.** Simple role *is a role which satisfies at least one of the following conditions:*

(a) role name which does not occur on a right-hand side of any RIA, or

(b) R^- *where* R *is a simple role, or*

(c) role name occuring on a right-hand side of RIA if every such RIA has only one simple role on its left-hand side.

All other roles are non-simple*.*

For description logic deciding, sound and complete tableau algorithms are usually used. It is out of the scope of this thesis to describe the tableau algorithm for $\mathcal{SROIQ}$, but a curious reader can find the details in work *The even more irresistible* $\mathcal{SROIQ}$ by Horrocks et al. (2006).

The following conditions form a sufficient condition for $\mathcal{SROIQ}$ decidability: only simple roles may appear in qualified number restrictions, self-restriction, role disjointness axioms and axioms in form $a, b \colon \neg R$; RBox is regular; universal role does not appear in any RIA nor in any role assertion.

Theorem 1. *If the above mentioned conditions hold, $\mathcal{SROIQ}$ is decidable (Horrocks et al., 2006) and* N2ExpTime*-complete (Kazakov, 2008).*

1.2 $\mathcal{SHOIQ}$ DL

$\mathcal{SHOIQ}$ is a fragment of $\mathcal{SROIQ}$, which does not feature the universal role U; role disjointness, reflexivity nor irreflexivity assertions; individual assertions of the form $a, b\colon \neg R$; complex role inclusion axioms with two or more different roles on the left-hand side; self-restrictions.

Note that $\mathcal{SHOIQ}$ features only those role assertions, which can be expressed via non-complex RIAs with one role on the left-hand side: $\text{Sym}(R)$ holds iff $R^- \sqsubseteq R$ holds and $\text{Tra}(R)$ holds iff $R \cdot R \sqsubseteq R$ holds.

Assuming unary coding of numbers in the input $\mathcal{SHOIQ}$ is NExpTime-complete, without the assumption it is NExpTime-hard (Tobies, 2001).

1.3 $\mathcal{ALCHOIQ}$ DL

$\mathcal{ALCHOIQ}$ is a fragment of $\mathcal{SHOIQ}$, which does not feature role transitivity assertion nor RIAs in form $R \cdot R \sqsubseteq R$. (It features only RIAs with one role on the left-hand side).

We already know that assuming unary coding of numbers, $\mathcal{SHOIQ}$ is NExpTime-complete. Additionaly, $\mathcal{ALCOIQ}$ (a fragment of $\mathcal{ALCHOIQ}$) is also NExpTime-complete (Tobies, 2000). Since $\mathcal{ALCHOIQ}$ is more expressive than $\mathcal{ALCOIQ}$ but less than $\mathcal{SHOIQ}$, assuming unary coding of numbers, $\mathcal{ALCHOIQ}$ is also NExpTime-complete.

1.4 $\mathcal{SHIQ}$ DL

$\mathcal{SHIQ}$ is a fragment of $\mathcal{SHOIQ}$, which does not feature nominals, i.e., concept expressions in form $\{a\}$. $\mathcal{SHIQ}$ is ExpTime-complete (Tobies, 2001).

Chapter 2

Desiderata for Higher-Order Description Logics

In this chapter we discuss the motivation behind our work and the desired properties of higher-order extensions of DLs. As the example domain on which we illustrate our ideas we chose domain similar to the one in the first chapter: the biological taxonomy.

Sections of this chapter deal with different aspects of *metamodelling*, namely domain metamodelling and full metamodelling of selected features of language (De Giacomo et al., 2011). *Domain metamodelling* allows to use concepts and roles as instances, thus enabling their classification and restriction just as if they were individuals. In contrast, *full metamodelling* allows to model with language operators, such as the subsumption relationship.

First, we describe what are higher orders and when they can be handy. Then, we explain the motivation behind full metamodelling of selected language operators and show some examples of its use. We conclude this chapter with a brief summary of modelling features which we find desirable.

2.1 Higher-Order Concepts and Roles

The biological taxonomy classifies individual organisms into various species, genera, classes, etc. An example taxonomy (skipping some of the taxonomical ranks) is showed in Figure 2.1: individuals melman and zarafa belong to the species Giraffa camelopardalis (giraffe). Individuals yogi and humphrey belong to the species Ursus arctos (brown bear). Further, Giraffa camelopardalis and Ursus arctos are species of class Mammalia (mammals), while Aptenodytes forsteri (emperor penguin) is a species of class Aves (birds). Both classes Mammalia and Aves are a part of kingdom Animalia (animals).

The dashed arrow represents the relationship of being an instance – *instantiation*,

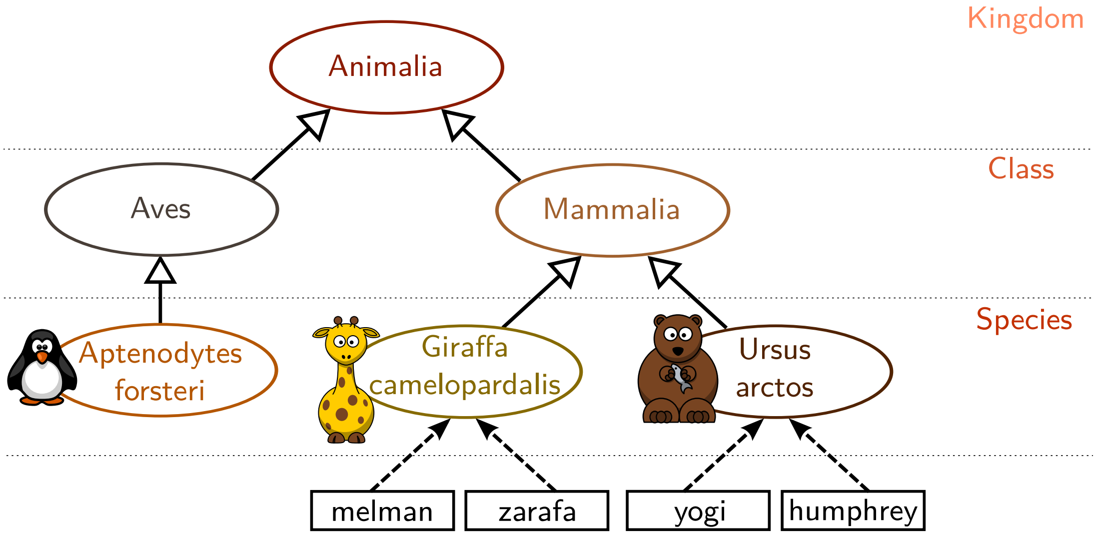

Figure 2.1: Example Taxonomy

and the solid arrow with white head represents the relationship of being a subclass (subconcept) – *subsumption*. E.g., melman is an instance of Giraffa camelopardalis, which is a subclass of Mammalia. Note that the instantiation arrow connects only individuals with concepts, while the subsumption arrow connects only concepts with concepts. This allows the taxonomy to be easily modelled in one of the standard description logics:

$$\begin{gathered} \mathsf{melman} : \mathsf{Giraffa\ camelopardalis} \\ \mathsf{zarafa} : \mathsf{Giraffa\ camelopardalis} \\ \mathsf{yogi} : \mathsf{Ursus\ arctos} \\ \mathsf{humphrey} : \mathsf{Ursus\ arctos} \\ \mathsf{Giraffa\ camelopardalis} \sqcup \mathsf{Ursus\ arctos} \sqsubseteq \mathsf{Mammalia} \\ \mathsf{Aptenodytes\ forsteri} \sqsubseteq \mathsf{Aves} \\ \mathsf{Mammalia} \sqcup \mathsf{Aves} \sqsubseteq \mathsf{Animalia} \end{gathered} \tag{2.1}$$

However, Figure 2.1 contains some information not captured by the DL formalization (2.1): the information that every concept directly above individuals is a species, every concept directly above species is a class and that the concept on the top of the figure is a kingdom. The problem is that the relationship between the concepts and their respective ranks is instantiation – e.g., Giraffa camelopardalis is an instance of Species, just as melman is an instance of Giraffa camelopardalis. Yet, this kind of instantiation is not allowed in standard DLs, because in standard DLs only individuals can act as instances. Figure 2.2 explicitly shows the instantiation relationship between different species, classes, kingdoms and the concepts Species, Class and Kingdom.

If we expressed the additional information from Figure 2.2 in a DL-like syntax, it

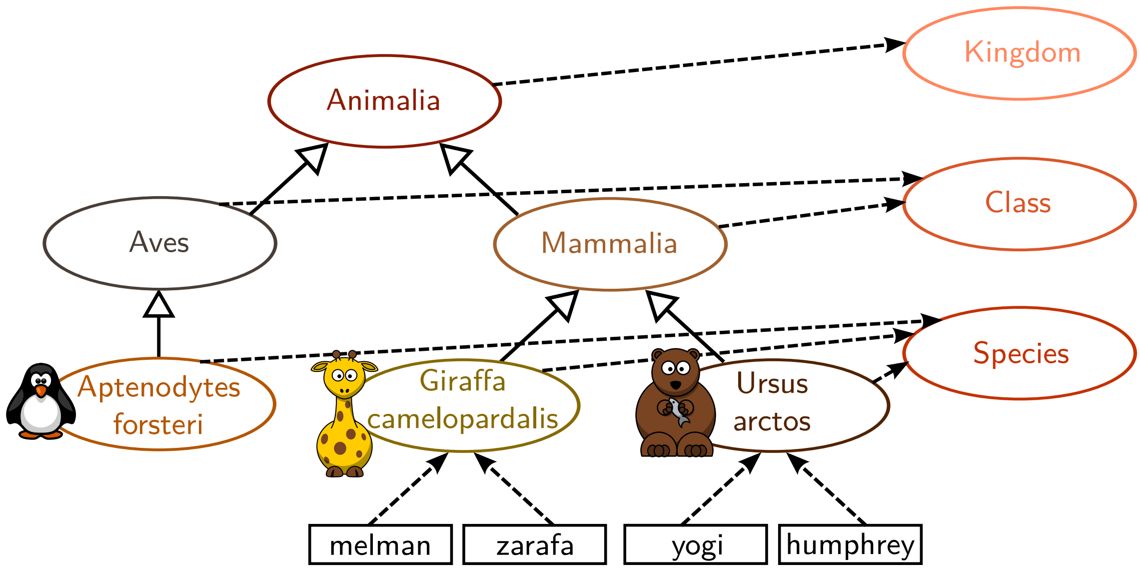

Figure 2.2: Example Taxonomy with Explicit Instantiation

would look as follows:

$$\begin{gathered}\mathsf{Giraffa\ camelopardalis}: \mathsf{Species}\\ \mathsf{Ursus\ arctos}: \mathsf{Species}\\ \mathsf{Aptenodytes\ forsteri}: \mathsf{Species}\\ \mathsf{Mammalia}: \mathsf{Class}\\ \mathsf{Aves}: \mathsf{Class}\\ \mathsf{Animalia}: \mathsf{Kingdom}\end{gathered} \tag{2.2}$$

Moreover, we can go even further when we realise that all species, classes, kingdoms (and other taxa) are instances of the concept Taxon and that Species, Class, Kingdom (and other taxonomical ranks) are instances of the concept Rank:

$$\begin{gathered}\mathsf{Species} \sqcup \mathsf{Class} \sqcup \mathsf{Kingdom} \sqsubseteq \mathsf{Taxon}\\ \mathsf{Species}: \mathsf{Rank}\\ \mathsf{Class}: \mathsf{Rank}\\ \mathsf{Kingdom}: \mathsf{Rank}\end{gathered} \tag{2.3}$$

In this ontology, Species, Class, Kingdom, Taxon and Rank are *higher-order concepts* – they classify concepts, not individuals. Note that while Species or Class classify first-order concepts, Rank classifies concepts which classify first-order concepts. Consequently, just as Giraffa camelopardalis and other species are first-order concepts, we will call Species and other ranks second-order concepts and Rank a third-order concept (see Figure 2.3).

We can continue with describing a *higher-order role*, i.e., a role which can interconnect instances with different orders (including individuals, which can be viewed as

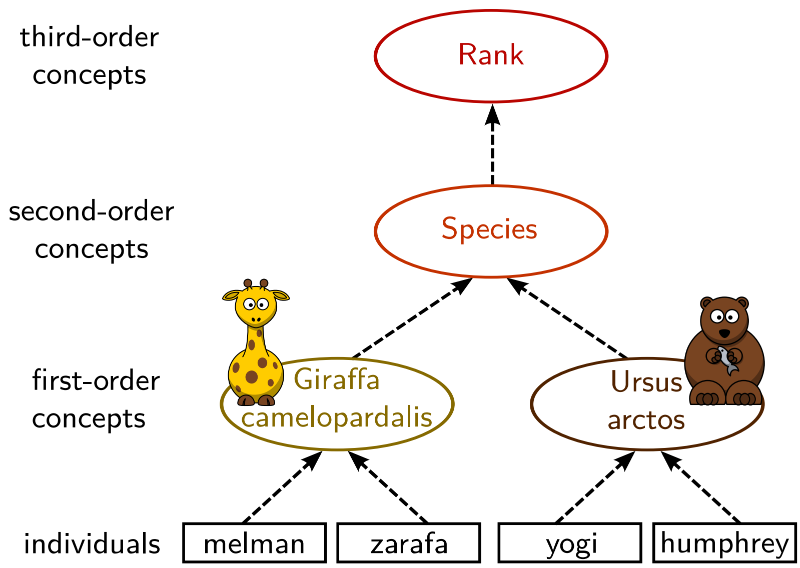

Figure 2.3: Concepts of Different Orders

zeroth-order objects). A good example is the role definedBy which relates taxa and ranks (first- and second-order concepts) with people (individuals) who defined them:

$$\begin{aligned} \exists \mathsf{definedBy}.\top &\sqsubseteq \mathsf{Taxon} \sqcup \mathsf{Rank} \\ \top &\sqsubseteq \forall \mathsf{definedBy}.\mathsf{Person} \\ \mathsf{Mammalia}, \mathsf{vonLinn\acute{e}} &: \mathsf{definedBy} \end{aligned} \tag{2.4}$$

Moreover, higher-order concepts could classify not only concepts and individuals, but also roles, thus enabling us to systematically classify roles into a hierarchy, e.g., to classify different kinds of animal behaviour:

$$\begin{aligned} \mathsf{migratesTo} &: \mathsf{Behaviour} \\ \mathsf{imitates} &: \mathsf{LearningBehaviour} \\ \mathsf{LearningBehaviour} &\sqsubseteq \mathsf{Behaviour}. \end{aligned} \tag{2.5}$$

2.2 Instantiation and Subsumption Metamodelling

Having concepts, such as species, classified in higher-order concepts, such as Species, we might want to characterize individuals of all species with some feature, or, vice versa, to assert properties of species based on features of their individuals. A statement of the former kind is "individuals of all species are organisms". An example of the latter kind of statements is "a living organism's species is not extinct". These statements traverse the orders, i.e., classify or restrict concepts based on properties of instances of their instances and vice versa.

Since the orders are connected through the instantiation relationship, an intuitive way of traversing the orders would be to have a role which connects instances with concepts which classify them (corresponding to the dashed arrow). With this role, which we will name instanceOf, we could express axioms about concepts of different orders, just like the ones from the previous paragraph (2.6), (2.7). Moreover, instanceOf allows us to traverse orders in both directions in one complex concept, e.g., a concept classifying all individuals of the same species as yogi, without having to know what yogi's species is (2.8).

$$\exists \mathsf{instanceOf}.\mathsf{Species} \sqsubseteq \mathsf{Organism} \tag{2.6}$$

$$\mathsf{Species} \sqcap \exists \mathsf{instanceOf}^{-}.\mathsf{Alive} \sqsubseteq \neg \mathsf{Extinct} \tag{2.7}$$

$$\exists \mathsf{instanceOf}.(\mathsf{Species} \sqcap \exists \mathsf{instanceOf}^{-}.\{\mathsf{yogi}\}) \tag{2.8}$$

After modelling with the instantiation relationship a natural next step is subsumption. Similarly to the instantiation, we will model it by a role with name subClassOf (corresponding to the solid arrow with white head). Such a role has analogous usage to the role instanceOf, but instead of traversing the orders of the ontology, it makes obtaining the sub- or superconcepts possible. An example of use is the axiom

$$\mathsf{Species} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Kingdom}, \tag{2.9}$$

expressing that each species is a subclass of some kingdom.

Note that modelling with instanceOf and subClassOf is a part of the full metamodelling mentioned in the introduction to this chapter – it allows us to model with the relationships of instantiation and subsumption, expressed also by the language operators : and $\sqsubseteq$, just as with any other role. While the relationships expressed by axioms featuring instanceOf and subClassOf could be expressed also without metamodelling, it would be necessary to explicitly state some facts that are implicit in the axioms with instanceOf and subClassOf.

Instead of axiom (2.6) we could use axioms Aptenodytes forsteri $\sqsubseteq$ Organism, Giraffa camelopardalis $\sqsubseteq$ Organism and Ursus arctos $\sqsubseteq$ Organism, which would be sufficient in the current state of our ontology, but in case we later added another species (e.g., Bufo bufo, the common toad), we would also have to add another axiom (Bufo bufo $\sqsubseteq$ Organism). Axiom (2.7) cannot be expressed straightforwardly without the role instanceOf and without knowing which species have at least one living specimen, although there certainly are some elaborate ways to do it (see Section 4.3). Similarly to the first axiom, instead of axiom (2.8) we could just use the concept corresponding to yogi's species, which in this case is Ursus arctos – which is, clearly, possible only when we know what yogi's species is. Finally, instead of axiom (2.9) we would have to classify each species as a subconcept of some kingdom, which is, in our case, already done by axioms (2.1). However, in general we have the same problem as with axiom (2.6) – for each new species we would have to add a new axiom.

2.3 Summary of Metamodelling Features

In the previous sections of this chapter we have introduced and motivated two kinds of metamodelling, which we find beneficial. The first one is domain metamodelling achieved through higher-order concepts and roles. The second one is full metamodelling of the instantiation and subsumption relationship. (Exploring metamodelling of other language operators is out of scope of this thesis, however interesting it might be.)

Since these metamodelling features are not naturally present in the standard description logics, in the rest of this thesis we focus on their analysis. In the following chapter, we introduce selected works on higher-order description logics, and in Chapter 4 we present our approach to this interesting problem.

Chapter 3

Related Work

In this chapter we introduce several works related to the topic of higher-order description logics. All of them (except OWL) explore the possibilities of domain metamodelling – i.e., modelling with concepts and roles as if they were individuals. First, in Section 3.1 we describe a higher-order logic HiLog, originally motivated by higher-order logic programming, both used as basis for semantics of several higher-order description logics. Then, in Section 3.2 we introduce OWL, a family of modelling languages partially based on description logics. Finally, in Sections 3.3 to 3.9 we describe several higher-order description logics with various metamodelling capabilities.

In our work presented in the next chapter we reuse selected ideas from $\mathcal{O}^{\text{meta}}$ (Glimm et al., 2010) and $\mathcal{TH}(\mathcal{SROIQ})$ (Homola et al., 2014). A comparison of related work with our approach, including a discussion on selected properties, follows in Chapter 5.

3.1 HiLog

HiLog is a logic introduced by Chen et al. (1993). The motivation behind it was to support higher-order and meta-level programming constructions with higher-order syntax in logic programming, while keeping the semantics first-order. This results in sound and complete inference based on resolution. Additionally, HiLog is reducible in the first-order predicate logic.

3.1.1 Syntax

In contrast with the first-order logic, HiLog syntax uses in addition to auxiliary symbols, logical connectives and quantifiers only one set $\mathcal{S}$ of *parameter symbols*. Any parameter symbol can be used as either constant, function, or predicate. Moreover, in HiLog there is no such thing as arity – any symbol can be used instead of t in the term $t(t_1, \ldots, t_n)$ for arbitrary $n \geq 1$. Terms are defined similarly to FOL terms. However, an atomic formula (atom) is any term. Complex formulas are build from the atoms using the usual logical connectives (¬, ∧, ∨, ∀ and ∃).

3.1.2 Semantics

A semantic structure for HiLog is a tuple consisting of:

- a set of *intensions* U,
- a set $U_{true} \subseteq U$ specifying which $u \in U$ are intensions of true propositions,
- a function I assigning each parameter symbol $s \in \mathcal{S}$ an intension $I(s) \in U$,
- a function $\mathcal{F}$ assigning each intension $u \in U$ its *extensions*: a function from $U^k \to U$ for each $k \geq 1$.

Each symbol $s \in \mathcal{S}$ is associated with one element of U, which act as s's representative. This element of U determines its truth value (corresponding to the interpretation of s as a predicate symbol in FOL) as well as its interpretation when used as a function symbol with an arbitrary arity. The arguments $t_1, \ldots, t_n$ of a function interpreting s used as a n-ary funcition symbol are evaluated as their intensions: $I(t_1), \ldots, I(t_n)$.

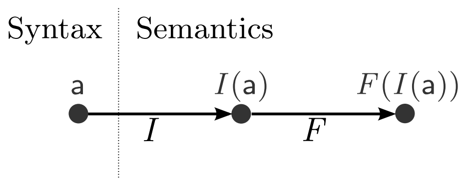

Figure 3.1: HiLog Syntax and Semantics

Hence, the interpretation is two-step (see Figure 3.1): first, each syntactic element is assigned its intension. Then, each intension is assigned its extension or truth value. Thus, while the syntax is higher-order, the semantics is first-order. In the following, *HiLog-based semantics* will refer to such a two-step semantics.

3.2 OWL and OWL 2

OWL (Hayes et al., 2004) and OWL 2 (W3C OWL Working Group, 2012) are knowledge representation languages. OWL is a family of three languages which vary in expressivity: OWL Lite is the least expressive one, OWL DL is more expressive, but still decidable and OWL Full is the most expressive, but undecidable (Motik, 2007). While OWL Lite and OWL DL are first-order, OWL Full allows full metamodelling of all its constructors, which is the source of its undecidability. Both OWL Lite and OWL

DL are based on description logics – $\mathcal{SHIF}$ (a fragment of $\mathcal{SHIQ}$) and $\mathcal{SHOIN}$ (a fragment of $\mathcal{SHOIQ}$), respectively. OWL 2 is more expressive than OWL DL, yet still decidable. It is based on $\mathcal{SROIQ}$.

In the OWL area, "class" is often used instead of "concept" and "property" is often used instead of "role". Due to its practical nature and roots in older knowledge representation languages, there are several syntaxes for OWL. For the sake of intelligibility, we will use the syntax of the respective underlying DL when describing a work extending OWL.

3.3 OWL FA

OWL FA is an extension of OWL DL proposed by Pan et al. (2005). Syntactically as well as semantically, higher-order classes and properties are divided in multiple *strata*. Class (property) of the ith stratum can classify (interconnect) only elements from the $(i-1)$th stratum. Classes (properties) belonging to the first stratum can classify (interconnect) only individuals.

Semantics of OWL FA is higher-order. The domain is a disjoint union of domains for individual strata, where the domain for $(i+1)$th stratum $\Delta^{\mathcal{I}}_{i+1}$ consists of the power set of the domain for the ith stratum $2^{\Delta^{\mathcal{I}}_i}$ and the power set of the cartesian square of the domain for the ith stratum $2^{\Delta^{\mathcal{I}}_i \times \Delta^{\mathcal{I}}_i}$. Classes of ith stratum are interpreted as subsets of the domain for ith stratum, and properties of the ith stratum as relations on the domain for ith stratum. Pan et al. (2005) proved that the problem of OWL FA knowledge base satisfiability is reducible in a finite number of steps to the problem of OWL DL knowledge base satisfiability.

OWL FA supports domain metamodelling as described in Section 2.1, but it does not support full metamodelling of any language operator.

3.4 Motik's π-Semantics and ν-Semantics

Motik (2007) not only proved OWL Full undecidable, but also proposed two different semantics under which are both $\mathcal{ALCHOIQ}$ and $\mathcal{SHOIQ}$ extended with syntactic metamodelling features decidable (however, $\mathcal{SHOIQ}$ only under technical assumption that each two role names are distinct). Syntax used in Motik's work differs from the standard syntax of $\mathcal{ALCHOIQ}$ and $\mathcal{SHOIQ}$ by using only one set of names instead of separating them in the set of individual names, concept names and role names. This way can any name act as an individual, concept and role.

3.4.1 π-Semantics

Under *π-semantics* (also called *contextual*) every symbol is assigned not only an intension but also two extensions – concept and role extension. The extensions do not depend on the intension, so it is possible for two objects with the same intension to have different extensions. (Note that this might often be an undesired characteristic. For an example see Section 5.1.)

Since the semantics is only seemingly higher-order, it can be decided using known algoritms for $\mathcal{ALCHOIQ}$ and $\mathcal{SHOIQ}$. Thus, knowledge base satisfiability of both $\mathcal{ALCHOIQ}$ and $\mathcal{SHOIQ}$ under π-semantics is decidable in NExpTime.

Motik's π-semantics supports domain metamodelling only to a small extent and it does not support full metamodelling at all.

3.4.2 ν-Semantics

ν-semantics is HiLog-based – every name (but not complex concepts or roles) is assigned an intension and every intension is assigned both concept extension and role extension. This leads to *intensional regularity*: if the intensions of two symbols are equal, so are their extensions.

Motik showed that satisfiability of a $\mathcal{ALCHOIQ}$ knowledge base under ν-semantics can be exponentially reduced to knowledge base satisfiability under π-semantics, resulting in NExpTime algorithm. In case of a $\mathcal{SHOIQ}$ knowledge base, intensional regularity makes the situation more complicated. To make the problem of knowledge base satisfiability under the ν-semantics decidable even if it contains transitivity statements (recall that the difference between $\mathcal{ALCHOIQ}$ and $\mathcal{SHOIQ}$ is that the latter allows transitivity assertions), the knowledge base must employ the *unique role assumption* (URA), i.e., it must contain an unequality axiom for each two names used as roles. With URA, deciding knowledge base satisfiability under the ν-semantics can be polynomially reduced to the satisfiability of $\mathcal{ALCHOIQ}$ knowledge base under the ν-semantics.

Motik's ν-semantics supports domain metamodelling via higher orders, but it does not support full metamodelling of any language operator.

3.5 Punning in OWL 2

Punning is an OWL 2 feature introduced by Cuenca Grau et al. (2008). It allows to use any name as individual, class and property at the same time. The semantics corresponds to the π-semantics (Motik, 2007), making the interpretations of the name

Table 3.1: Axiomatization of the Second Order (Glimm et al., 2010)

$\mathsf{Inst} \equiv \neg\mathsf{Class}$	
$\mathsf{o}_A : \mathsf{Class}$	for each $A \in N_\mathrm{C}$
$\mathsf{a} : \mathsf{Inst}$	for each $a \in N_\mathrm{I}$
$\exists R.\top \sqsubseteq \mathsf{Inst}$	for each $R \in N_\mathrm{R}$
$\top \sqsubseteq \forall R.\mathsf{Inst}$	for each $R \in N_\mathrm{R}$
$\exists\mathsf{type}.\top \sqsubseteq \mathsf{Inst}$	
$\top \sqsubseteq \forall\mathsf{type}.\mathsf{Class}$	
$\exists\mathsf{subClassOf}.\top \sqsubseteq \mathsf{Class}$	
$\top \sqsubseteq \forall\mathsf{subClassOf}.\mathsf{Class}$	
$A \equiv \exists\mathsf{type}.\{\mathsf{o}_A\}$	for each $A \in N_\mathrm{C}$
$\mathsf{Class} \sqcap \forall\mathsf{type}^{-}.\exists\mathsf{type}.\{\mathsf{o}_A\} \equiv \mathsf{Class} \sqcap \exists\mathsf{subClassOf}.\{o_A\}$	for each $A \in N_\mathrm{C}$

in different contexts independent (hence the name "punning") and thus making OWL 2 with punning decidable in the same time as OWL 2. Further, it means that punning's support of domain metamodelling is limited and it does not support full metamodelling at all.

3.6 Ontology-Inherent Metamodelling: $\mathcal{O}^{\text{meta}}$

Glimm et al. (2010) proposed an encoding scheme enabling second-order metamodelling in OWL 2. It allows a first-order ontology $\mathcal{O}$ to be extended to second-order ontology $\mathcal{O}^{\text{meta}}$ by means of encoding it into first-order vocabulary and axioms with first-order syntax and semantics. The basic vocabulary $N = (N_\mathrm{I}, N_\mathrm{C}, N_\mathrm{R})$ of $\mathcal{O}$ is extended by new individual names o_A for each $A \in N_\mathrm{C}$ (which act as representatives for A), new class names Inst (classifying individuals from N_I) and Class (classifying the new individuals) and new roles type, $\mathsf{subClassOf}$ and R_{Inst}. Axiomatization of $\mathcal{O}^{\text{meta}}$ can be found in Table 3.1. Additionally, all axioms from $\mathcal{O}$ are modified to ensure that the domain of $\mathcal{O}$ and class constructions are contained in Inst.

If we want to assert that a first-order class A belongs to a second-order class C, we use axiom $\mathsf{o}_A : C$. Similarly, if we want to describe a nominal classifying A, we use $\{\mathsf{o}_A\}$. All roles from N_R are first-order, interconnecting only original individuals (axioms in form $\mathsf{o}_A, \mathsf{o}_B : R$ are not allowed). Role type is axiomatized to connect each a with each o_C such that $a^\mathcal{I} \in C^\mathcal{I}$. Similarly, role $\mathsf{subClassOf}$ is axiomatized to connect each o_A with each o_B such that $A^\mathcal{I} \subseteq B^\mathcal{I}$. Role R_{Inst} is introduced for technical purposes.

Since $\mathcal{O}^{\text{meta}}$ is encoded in the first-order ontology, it can be decided by known algorithms for OWL 2, with $\mathcal{SROIQ}$ time complexity N2ExpTime. The authors proved some key properties of their approach, but they did not provide its full semantic characterization.

$\mathcal{O}^{\text{meta}}$ partially allows domain metamodelling (there are only two orders) and also allows full metamodelling of instantiation and subsumption.

3.7 $Hi(\mathcal{SHIQ})$

De Giacomo et al. (2011) presented the syntax and semantics of extension applicable to any DL $\mathcal{L}$ allowing higher-order modeling, called $Hi(\mathcal{L})$. The proposed syntax is higher-order (and similar to the syntax proposed by Motik since it only contains one set of names), however, semantics is HiLog-based, assigning each symbol its intension (called domain object) and assigning each intension its extensions – for both roles and concepts, including complex concepts. This is done by interpreting language operators as functions on intensions.

$Hi(\mathcal{L})$ is further examined for $\mathcal{L} = \mathcal{SHIQ}$. The authors propose a translation Π from $Hi(\mathcal{SHIQ})$ to the first-order $\mathcal{SHIQ}$ such that $Hi(\mathcal{SHIQ})$ knowledge base $\mathcal{K}$ is satisfiable iff $\mathcal{SHIQ}$ knowledge base $\Pi(\mathcal{K})$ is satisfiable. Using this translation is the satisfiability of $Hi(\mathcal{SHIQ})$ knowledge base ExpTime-complete.

$Hi(\mathcal{SHIQ})$ supports domain metamodelling via higher orders, but it does not support full metamodelling of any language operator.

3.8 $\mathcal{TH}(\mathcal{SROIQ})$

$\mathcal{TH}(\mathcal{SROIQ})$ is a metamodelling approach proposed by Homola et al. (2014). Syntactically, $\mathcal{TH}(\mathcal{SROIQ})$ is a typed higher-order extension of $\mathcal{SROIQ}$. The typing is similar to the OWL FA stratification: each symbol has some type assigned. In case of concepts it is a number $t > 0$, for roles it is a pair of numbers $t, u > 0$ (because roles connect two objects) and all individuals have type 0.

The vocabulary of $\mathcal{TH}(\mathcal{SROIQ})$ consists of disjoint sets of concept names and role names. The set of concept (role) names is itself decomposed into disjoint sets of concept (role) names N_C^t (N_R^{tu}) of type t, where $t > 0$ (tu, where $t, u > 0$). The set of individual names N_I is also denoted N_C^0.

Individual assertions of concept membership in $\mathcal{TH}(\mathcal{SROIQ})$ can only express that a element of type $t-1$ is a member of a concept of type t. Similarly, assertion of role membership can only express that elements of type $t-1$ and $u-1$ are related by a role of type tu. GCIs (RIAs and role disjointness assertions) can state only relationships between concepts (roles) of the same type $t > 0$ (or tu for $t, u > 0$). Role reflexivity assertion can be applied only to roles with type tt for $t > 0$.

The semantics of $\mathcal{TH}(\mathcal{SROIQ})$ is HiLog-based, assigning a typed concept intension to each concept or individual name and a typed role intension to each role name. For intensions with type $t > 0$ or tu such that $t, u > 0$ is also defined their extension – a set of intensions of type $t - 1$ for intensions of type t and a relation on intensions of type $(t-1)(u-1)$ for intensions of type tu. Note that the semantics of $\mathcal{TH}(\mathcal{SROIQ})$ is intensionally regular.

However, despite the fact that $\mathcal{TH}(\mathcal{SROIQ})$ is based on $\mathcal{SROIQ}$, $\mathcal{TH}(\mathcal{SROIQ})$ is not more expressive than Motik's extension of $\mathcal{SHOIQ}$ nor $Hi(\mathcal{SHIQ})$, because in $\mathcal{TH}(\mathcal{SROIQ})$ concepts cannot classify instances of different types. E.g., it would not be possible to use one role $\mathsf{definedBy}$ to interconnect both pairs $(\mathsf{Mammalia}, \mathsf{von\ Linné})$ and $(\mathsf{Class}, \mathsf{de\ Tournefort})$, because $\mathsf{Mammalia}$ is a first-order concept, while Class is a second-order concept.

Homola et al. (2014) proved that the problem of satisfiability of a $\mathcal{TH}(\mathcal{SROIQ})$ knowledge base is polynomially reducible to the problem of $\mathcal{SROIQ}$ knowledge base satisfiability. Thus, concept satisfiability and subsumption in $\mathcal{TH}(\mathcal{SROIQ})$ are decidable in N2ExpTime.

$\mathcal{TH}(\mathcal{SROIQ})$ supports domain metamodelling via higher orders, but it does not support full metamodelling of any language operator.

3.9 $\mathcal{SHIQM}$

$\mathcal{SHIQM}$ is a metamodelling extension of $\mathcal{SHIQ}$ proposed by Motz et al. (2015). It extends the standard $\mathcal{SHIQ}$ knowledge base with a new part – *MBox* containing *metamodelling axioms* in form $a =_{\mathrm{m}} A$ where a is an individual and A is an atomic concept. The metamodelling axioms have meaning similar to the axiom $A \equiv \exists \mathsf{type}.\mathsf{o}_A$ in $\mathcal{O}^{\mathrm{meta}}$: by asserting $a =_{\mathrm{m}} A$, we state that a is a representative of A.

The semantics is higher-order – the domain is a subset of S_n for some n, where S_0 is a set of atomic objects, and $S_{i+1} = S_i \uplus 2^{S_i}$ for each $i \geq 0$. Thus, the domain contains basic objects, sets of basic objects, sets of sets of basic objects etc. These elements of the domain are not separated in any way – any of them can serve as the interpretation of an individual or a concept. In fact, axiom $a =_{\mathrm{m}} A$ requires that $a^{\mathcal{I}} = A^{\mathcal{I}}$. Thus, if the representatives of two concepts are equal, also the concepts' extensions have to be equal ($\mathcal{SHIQM}$ has the property of intensional regularity), and vice versa ($\mathcal{SHIQM}$ has also the property of *extensionality*).

Motz et al. (2015) proved the decidability of $\mathcal{SHIQM}$ by extending the tableau algorithm for $\mathcal{SHIQ}$, but they did not mention the time complexity. However, $\mathcal{ALCQM}$

(a fragment of $\mathcal{SHIQM}$) is believed to be ExpTime-complete (Motz et al., 2014), just as $\mathcal{ALCQ}$ (afragment of $\mathcal{SHIQ}$), its underlying description logic.

$\mathcal{SHIQM}$ supports domain metamodelling via higher orders, but it does not support full metamodelling of any language operator.

Chapter 4

Higher-Order DLs $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$

In this chapter we describe our approach to higher-order description logics supporting domain metamodelling, full metamodelling of instantiation, and partially also full metamodelling of subsumption. Our approach is influenced mainly by Homola et al. (2014) and Glimm et al. (2010).

Given a description logic $\mathcal{L}$, we introduce four higher-order extensions for it: logic $\mathcal{HIR}(\mathcal{L})$ (pronounced as "higher") and logics $\mathcal{HIRS}_{\text{NN}}(\mathcal{L})$, $\mathcal{HIRS}_{\text{NA}}(\mathcal{L})$ and $\mathcal{HIRS}_{\text{SN}}(\mathcal{L})$, together dubbed $\mathcal{HIRS}_*(\mathcal{L})$ (pronounced as "highers"), which further extend $\mathcal{HIR}(\mathcal{L})$. They all feature:

1. higher-order concepts called *meta-concepts* and higher-order roles called *meta-roles*. Both can have not only individuals, but also concepts (designated by the letter $\mathcal{H}$) and roles (designated by the letter $\mathcal{R}$) as instances,

2. role instanceOf which metamodels the relationship between instances and the concepts which classify them (designated by the letter $\mathcal{I}$).

Additionaly, logics $\mathcal{HIRS}_*(\mathcal{L})$ feature:

3. role subClassOf which metamodels the relationship between two concept where the second subsumes the first one (designated by the letter $\mathcal{S}$). Different variants of $\mathcal{HIRS}_*(\mathcal{L})$ offer different properties of the subClassOf role.

Higher-order concepts and roles are enabled by relaxing the syntactic constraints on instances: $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ allow to use any name as an instance in both nominals and in the left-hand side of individual assertions. Roles instanceOf and subClassOf are a fixed part of a $\mathcal{HIR}(\mathcal{L})$ or $\mathcal{HIRS}_*(\mathcal{L})$ vocabulary. We formally introduce $\mathcal{HIR}(\mathcal{L})$ syntax in Section 4.1, and $\mathcal{HIRS}_*(\mathcal{L})$ syntax in subsections of Section 4.4 (each $\mathcal{HIRS}_*(\mathcal{L})$ logic has its own subsection).

$\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ employ HiLog-based semantics (see Subsection 3.1.2). Names are first assigned domain objects (intensions), which act as representatives of the corresponding names. Then, concept and role intensions are assigned extensions: sets of intensions for concepts, or sets of pairs of intensions for roles. More about $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ semantics can be found in Section 4.2 and subsections of Section 4.4.

In Section 4.3, we show the decidability of $\mathcal{HIR}(\mathcal{L})$. Proofs of decidability for $\mathcal{HIRS}_*(\mathcal{L})$ logics can be found in the respective subsections of Section 4.4.

In $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$, concepts can classify and roles can interconnect a mix of different entities, e.g., a concept can classify individuals, concepts and roles. This property is called *promiscuity*. Conversely, if desired, concepts and roles can be axiomatized such that they classify and interconnect only some types of entities, e.g., a role can be axiomatized to interconnect only individuals with concepts classifying individuals. We show the axiomatization in Section 4.5.

In the course of the chapter, we show how $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ fulfill the requirements from Chapter 2. Further, a discussion on properties of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ as well as comparison with other logics allowing domain or full metamodelling can be found in the following chapter.

4.1 Syntax of $\mathcal{HIR}(\mathcal{L})$

$\mathcal{HIR}(\mathcal{L})$ syntax is built on the syntax of the description logic $\mathcal{L}$. On top of all standard features of $\mathcal{L}$ it allows concepts and roles in places where only individuals could be in the standard description logics. Also, $\mathcal{HIR}(\mathcal{L})$ vocabulary contains the role $\mathsf{instanceOf}$.

The following definition introduces the syntax of $\mathcal{HIR}(\mathcal{SROIQ})$. For any fragment $\mathcal{L}$ of the logic $\mathcal{SROIQ}$, the syntax of $\mathcal{HIR}(\mathcal{L})$ is the corresponding fragment of $\mathcal{HIR}(\mathcal{SROIQ})$.

Definition 4.1 ($\mathcal{HIR}(\mathcal{SROIQ})$ syntax)**.** *Let $N = N_{\mathrm{I}} \uplus N_{\mathrm{C}} \uplus N_{\mathrm{R}}$ be a description logic vocabulary such that* $\mathsf{instanceOf} \in N_{\mathrm{R}}$*.* $\mathcal{HIR}(\mathcal{SROIQ})$ role expressions *are inductively defined as the smallest set containing the expressions listed in Table 4.1 (upper part), where $R_0 \in N_{\mathrm{R}} \setminus \{\mathsf{instanceOf}, \mathsf{U}\}$, R is an atomic or inverse role, S and Q are role expressions.*

An expression is a $\mathcal{HIR}(\mathcal{SROIQ})$ concept *if it is of one of the forms listed in Table 4.1 (middle part), where $A \in N_{\mathrm{C}}$, $B \in N$, C and D are concpets, and R is an atomic or inverse role.*

Table 4.1: Syntax and Semantics of $\mathcal{HIR}(\mathcal{SROIQ})$ Expressions and Axioms

Syntax (x)	$\mathcal{HIR}(\mathcal{SROIQ})$ extension ($x^{\mathcal{E}}$)
R_0	$R_0^{\mathcal{IE}}$
R^-	$\{ (y,x) \mid (x,y) \in R^{\mathcal{E}} \}$
U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
$R_1 \cdots R_n$	$R_1^{\mathcal{E}} \circ \cdots \circ R_n^{\mathcal{E}}$
$\mathsf{instanceOf}$	$\{ (x,y) \mid x \in \Delta^{\mathcal{I}} \wedge y \in \Delta_{\mathrm{C}}^{\mathcal{I}} \wedge x \in y^{\mathcal{E}} \}$
A	$A^{\mathcal{IE}}$
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{E}}$
$C \sqcap D$	$C^{\mathcal{E}} \cap D^{\mathcal{E}}$
$\{B\}$	$\{B^{\mathcal{I}}\}$
$\exists R.C$	$\{ x \mid \exists y.(x,y) \in R^{\mathcal{E}} \wedge y \in C^{\mathcal{E}} \}$
$\geqslant nR.C$	$\{ x \mid \#\{ y \mid (x,y) \in R^{\mathcal{E}} \wedge y \in C^{\mathcal{E}} \} \geq n \}$
$\exists R.\mathsf{Self}$	$\{ x \mid (x,x) \in R^{\mathcal{E}} \}$
Axiom φ	Semantic constraint
$C \sqsubseteq D$	$C^{\mathcal{E}} \subseteq D^{\mathcal{E}}$
$B \colon C$	$B^{\mathcal{I}} \in C^{\mathcal{E}}$
$B_1, B_2 \colon R$	$(B_1^{\mathcal{I}}, B_2^{\mathcal{I}}) \in R^{\mathcal{E}}$
$B_1, B_2 \colon \neg R$	$(B_1^{\mathcal{I}}, B_2^{\mathcal{I}}) \notin R^{\mathcal{E}}$
$w \sqsubseteq R$	$w^{\mathcal{E}} \subseteq R^{\mathcal{E}}$
$\mathrm{Dis}(R,P)$	$R^{\mathcal{E}} \cap P^{\mathcal{E}} = \emptyset$

A $\mathcal{HIR}(\mathcal{SROIQ})$ knowledge base $\mathcal{K}$ *is a finite set of* axioms *of the forms listed in Table 4.1 (bottom part), where* $B, B_1, B_2 \in N$, C *and* D *are concepts,* R *and* P *are atomic or inverse roles, and* w *is a role chain.*

$\mathcal{HIR}(\mathcal{SROIQ})$ easily formalizes the taxonomic example from Chapter 2. Concept names can be used as instances in individual assertions, and thus taxa can be classified to meta-concepts of ranks (Species, Genus, Class, ...) (4.1) and ranks to the meta-meta-concept Rank (4.2). Meta-concepts are freely usable in GCIs just as concepts (4.3).

$$\begin{gathered}\mathsf{Giraffa\ camelopardalis} \colon \mathsf{Species}\\ \mathsf{Giraffa} \colon \mathsf{Genus}\\ \mathsf{Mammalia} \colon \mathsf{Class}\end{gathered} \tag{4.1}$$

$$\begin{gathered}\mathsf{Species} \colon \mathsf{Rank}\\ \mathsf{Genus} \colon \mathsf{Rank}\\ \mathsf{Class} \colon \mathsf{Rank}\end{gathered} \tag{4.2}$$

$$\begin{gathered}\mathsf{Species} \sqcup \mathsf{Genus} \sqcup \mathsf{Class} \sqcup \cdots \sqsubseteq \mathsf{Taxon} \\ \mathsf{Species} \sqcap \mathsf{Genus} \sqsubseteq \bot\end{gathered} \tag{4.3}$$

Meta-roles, such as definedBy already mentioned in Chapter 2 allow to associate not only individuals with individuals, but also concepts with individuals, e.g., a taxon or rank with a person (4.4), and also with other concepts, e.g., one species with its evolutionary successor species (4.5). We can then express complex meta-concepts such as LinneanSpecies (4.6).

$$\begin{gathered}\exists \mathsf{definedBy}.\top \sqsubseteq \mathsf{Taxon} \sqcup \mathsf{Rank} \\ \top \sqsubseteq \forall \mathsf{definedBy}.\mathsf{Person} \\ \mathsf{Mammalia}, \mathsf{vonLinn\acute{e}} : \mathsf{definedBy}\end{gathered} \tag{4.4}$$

$$\begin{gathered}\exists \mathsf{successorOf}.\top \sqsubseteq \mathsf{Species} \\ \top \sqsubseteq \forall \mathsf{successorOf}.\mathsf{Species}\end{gathered} \tag{4.5}$$

$$\mathsf{LinneanSpecies} \equiv \mathsf{Species} \sqcap \exists \mathsf{definedBy}.\{\mathsf{vonLinn\acute{e}}\} \tag{4.6}$$

Since $\mathcal{HIR}(\mathcal{SROIQ})$ allows roles as instances, we can also easily classify different types of animal behaviour:

$$\begin{gathered}\mathsf{migratesTo} : \mathsf{Behaviour} \\ \mathsf{imitates} : \mathsf{LearningBehaviour} \\ \mathsf{LearningBehaviour} \sqsubseteq \mathsf{Behaviour}\end{gathered} \tag{4.7}$$

First-order modelling still works as in $\mathcal{SROIQ}$: Individual organisms are classified to taxa and particular species are subsumed by their respective genera (4.8). Roles record that a specimen (a studied example individual of a species) was describedBy a person, and is locatedIn a museum (4.9).

$$\begin{gathered}\mathsf{zarafa} : \mathsf{Giraffa\ camelopardalis} \\ \mathsf{Giraffa\ camelopardalis} \sqsubseteq \mathsf{Giraffa}\end{gathered} \tag{4.8}$$

$$\begin{gathered}\mathsf{Specimen} \sqsubseteq \mathsf{Organism} \\ \exists \mathsf{describedBy}.\top \sqcup \exists \mathsf{locatedIn}.\top \sqsubseteq \mathsf{Specimen} \\ \top \sqsubseteq \forall \mathsf{describedBy}.\mathsf{Person} \sqcap \forall \mathsf{locatedIn}.\mathsf{Museum}\end{gathered} \tag{4.9}$$

4.2 Semantics of $\mathcal{HIR}(\mathcal{L})$

$\mathcal{HIR}(\mathcal{L})$ semantics is HiLog-based: Each name denotes a domain element (an intension) via the intension function $\cdot^{\mathcal{I}}$. The intensions for individuals, concepts, and roles are disjoint. Using the extension function $\cdot^{\mathcal{E}}$, concept intensions are assigned concept extensions (sets of intensions) and role intensions are assigned role extensions (relations on intensions). When a name is treated as a concept instance or a role actor, the semantics of a name is its intension. When treated as a concept or a role, the extension of the name's intension is considered.

The instanceOf role has fixed semantics: it connects every instance intension with the intension of each concept to whose extension it belongs.

Definition 4.2 ($\mathcal{HIR}(\mathcal{SROIQ})$ semantics)**.** *An* $\mathcal{HIR}(\mathcal{L})$ interpretation *of a DL vocabulary N with* instanceOf $\in N_\mathrm{R}$ *is a triple* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, \cdot^\mathcal{E})$ *such that:*

1. $\Delta^\mathcal{I} = \Delta^\mathcal{I}_\mathrm{I} \uplus \Delta^\mathcal{I}_\mathrm{C} \uplus \Delta^\mathcal{I}_\mathrm{R}$ *where* $\Delta^\mathcal{I}_\mathrm{I}, \Delta^\mathcal{I}_\mathrm{C}, \Delta^\mathcal{I}_\mathrm{R}$ *are pairwise disjoint,*

2. $a^\mathcal{I} \in \Delta^\mathcal{I}_\mathrm{I}$ *for each* $a \in N_\mathrm{I}$*,* $A^\mathcal{I} \in \Delta^\mathcal{I}_\mathrm{C}$ *for each* $A \in N_\mathrm{C}$*,* $R^\mathcal{I} \in \Delta^\mathcal{I}_\mathrm{R}$ *for each* $R \in N_\mathrm{R}$*,*

3. $R^\mathcal{I} \neq S^\mathcal{I}$ *whenever* R*,* $S \in N_\mathrm{R}$ *and* $R \neq S$ *(unique role assumption),*

4. $c^\mathcal{E} \subseteq \Delta^\mathcal{I}$ *for each* $c \in \Delta^\mathcal{I}_\mathrm{C}$*,* $r^\mathcal{E} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ *for each* $r \in \Delta^\mathcal{I}_\mathrm{R}$*.*

Extensions of role expressions $R^\mathcal{E}$ and of concepts $C^\mathcal{E}$ are inductively defined according to Table 4.1.

The above definition introduces the semantics of $\mathcal{HIR}(\mathcal{SROIQ})$. For any fragment $\mathcal{L}$ of the logic $\mathcal{SROIQ}$, the extensions of $\mathcal{HIR}(\mathcal{L})$ role expressions and concepts are inductively defined according to the corresponding fragment of Table 4.1.

Note that without the unique role assumption (URA, 3.) it is easy to cause two role names have the same intension (e.g., by an axiom $\{S\} \equiv \{T\}$), and thus also the same extension. That, as Motik (2007) has shown, leads to undecidability in logics admitting transitive roles and cardinality restrictions. The undecidability follows from the possibility of having two roles, S and T with the same intension (e.g., entailed by $\{S\} \equiv \{T\}$), where T is transitive (e.g., entailed by $\mathrm{Tra}(T)$). Syntactically, S is considered simple, (the equality of S's and T's intension can be entailed in a non-trivial way), and thus there is no syntactic problem with using it in a number restriction, even if it actually is transitive. This, as showed by Horrocks et al. (1999), leads to undecidability. (There is no similar problem requiring unique concept assumption, because we can assert the equality of two concept extensions directly in $\mathcal{SROIQ}$: $C \equiv D$.)

Satisfiability, model and entailment in $\mathcal{HIR}(\mathcal{L})$ are defined in the usual way.

Definition 4.3 ($\mathcal{HIR}(\mathcal{L})$ satisfiability)**.** *An axiom φ is* satisfied *by a $\mathcal{HIR}(\mathcal{L})$ interpretation $\mathcal{I}$ ($\mathcal{I} \models \varphi$) if $\mathcal{I}$ satisfies the respective semantic constraint from the lower part of Table 4.1.*

A $\mathcal{HIR}(\mathcal{L})$ interpretation $\mathcal{I}$ is a model *of $\mathcal{K}$ ($\mathcal{I} \models \mathcal{K}$) if $\mathcal{I}$ satisfies every axiom $\varphi \in \mathcal{K}$.*

A concept C is satisfiable in $\mathcal{K}$ *if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}} \neq \emptyset$.*

An axiom φ is entailed *by $\mathcal{K}$ ($\mathcal{K} \models \varphi$) if $\mathcal{I} \models \varphi$ holds for each $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$.*

The semantics of instanceOf should now be apparent. It allows to traverse meta-layers in modelling, just as we described in Chapter 2: Restrictions on instanceOf can select instances of concepts satisfying various meta-criteria, e.g., specimens of Linnean species described by someone else than von Linné (4.10). Conversely, restrictions on $\mathsf{instanceOf}^-$ select concepts whose instances satisfy complex criteria, e.g., species with specimens located in the British Museum (4.11). We can thus express that every instance of any taxon is an organism (4.12), i.e., that all taxa are effectively subconcepts of Organism. Assuming that every taxon is an instance of some rank and all ranks are instances of Rank, an equivalent statement is possible via the meta-meta-level (4.13). Applying a number restriction on instanceOf, we can also express mutual disjointness of Species by asserting that each Organism is classified as at most one species (4.14).

$$\begin{aligned} &\mathsf{Specimen} \sqcap \exists \mathsf{instanceOf}.(\mathsf{Species} \sqcap \exists \mathsf{definedBy}.\{\mathsf{vonLinn\acute{e}}\}) \\ &\qquad \sqcap \exists \mathsf{describedBy}.\neg\{\mathsf{vonLinn\acute{e}}\} \end{aligned} \tag{4.10}$$

$$\mathsf{Species} \sqcap \exists \mathsf{instanceOf}^-.(\mathsf{Specimen} \sqcap \exists \mathsf{locatedIn}.\{\mathsf{britishMusem}\}) \tag{4.11}$$

$$\exists \mathsf{instanceOf}.\mathsf{Taxon} \sqsubseteq \mathsf{Organism} \tag{4.12}$$

$$\exists \mathsf{instanceOf}.\exists \mathsf{instanceOf}.\mathsf{Rank} \sqsubseteq \mathsf{Organism} \tag{4.13}$$

$$\mathsf{Organism} \sqsubseteq \leqslant 1\, \mathsf{instanceOf}.\mathsf{Species} \tag{4.14}$$

Note that while Motik (2007) has suggested using SWRL rules (Horrocks and Patel-Schneider, 2004) to express axioms such as (4.13), in Section 4.3 we show that $\mathcal{SROIQ}$ is sufficiently expressive.

Liberal treatment of the instanceOf role allows creating its subroles, e.g., hasType assigning a prototypical specimen to each species (4.15), and using them in number restrictions, e.g., to assert that each species has exactly one holotype (the "most notable" specimen, usually used when the species was formaly described) and it is located in a major museum (4.16). While we could have created the meta-role hasType anyway, without using instanceOf we could not easily assure that it connects each species with one of its instances.

$$\begin{aligned} \mathsf{hasType} &\sqsubseteq \mathsf{instanceOf}^- \\ \exists \mathsf{hasType}.\top &\sqsubseteq \mathsf{Species} \\ \top &\sqsubseteq \forall \mathsf{hasType}.\mathsf{Specimen} \end{aligned} \tag{4.15}$$

$$\begin{aligned} \mathsf{Species} \sqsubseteq\; &\leqslant 1 \mathsf{hasType}.\mathsf{Holotype} \\ &\sqcap = 1 \mathsf{hasType}.(\mathsf{Holotype} \sqcap \exists \mathsf{locatedIn}.\mathsf{MajorMusem}) \end{aligned} \tag{4.16}$$

4.3 First-Order Reduction and Decidability

Now, we show how $\mathcal{HIR}(\mathcal{L})$ for some description logics $\mathcal{L}$ can be reduced to the first-order $\mathcal{L}$ (or, in some cases, $\mathcal{LO}$). We start with showing how $\mathcal{HIR}(\mathcal{SROIQ})$ can be reduced to $\mathcal{SROIQ}$. The reduction, fully defined below, is based on ideas by Glimm et al. (2010). A rough idea of how this reduction works can be found in the following lines.

For each concept A, a new individual name i_A is introduced to represent A's intension. Similarly, for each role R, a new "representative" individual name i_R is introduced. These new names representing concepts and roles are axiomatized to be instances of new concepts $\top_\mathrm{C}$ and $\top_\mathrm{R}$ of concept and role intensions. The relationship between the extension A and the intension i_A is expressed through the $\mathsf{instanceOf}$ role in the InstSync axioms. In the reduced knowledge base, $\mathsf{instanceOf}$ is an ordinary, axiomatized role. Since $\mathcal{HIR}(\mathcal{L})$ interpretations satisfy the unique role assumption (URA, Definition 4.2(3)), role intensions do not influence their extensions. Hence, there is no need for axiomatization of roles similar to the one in InstSync.

Definition 4.4 (First-Order Reduction for $\mathcal{HIR}(\mathcal{SROIQ})$)**.** *A DL vocabulary N with* $\mathsf{instanceOf} \in N_\mathrm{R}$ *is reduced into a DL vocabulary $N^1 := (N^1_\mathrm{C}, N^1_\mathrm{R}, N^1_\mathrm{I})$ where:*

- $N^1_\mathrm{C} = N_\mathrm{C} \uplus \{\top_\mathrm{C}, \top_\mathrm{R}\}$,
- $N^1_\mathrm{R} = N_\mathrm{R}$,
- $N^1_\mathrm{I} = N_\mathrm{I} \uplus \{\, \mathsf{i}_A \mid A \in N_\mathrm{C} \,\} \uplus \{\, \mathsf{i}_R \mid R \in N_\mathrm{R} \,\}$

for new names $\top_\mathrm{C}$, $\top_\mathrm{R}$, i_A and i_R for all $A \in N_\mathrm{C}, R \in N_\mathrm{R}$.

A given $\mathcal{HIR}(\mathcal{SROIQ})$ KB $\mathcal{K}$ in N is reduced into a $\mathcal{SROIQ}$ KB $\mathcal{K}^1 := \mathrm{Int}(\mathcal{K}) \cup \mathrm{InstSync}(N) \cup \mathrm{Typing}(N) \cup \mathrm{URA}(N)$ in N^1 where:

- $\mathrm{Int}(\mathcal{K})$ *is obtained from $\mathcal{K}$ by replacing each occurence of $A \in N_\mathrm{C}$ and $R \in N_\mathrm{R}$ in a nominal or in the left-hand side of a concept or (negative) role assertion by i_A and i_R, respectively.*
- $\mathrm{InstSync}(N)$ *consists of axioms $A \equiv \exists \mathsf{instanceOf}.\{\mathsf{i}_A\}$ for all $A \in N_\mathrm{C}$.*
- $\mathrm{Typing}(N)$ *consists of the following axioms for all $a \in N_\mathrm{I}$, $A \in N_\mathrm{C}$, and $R \in N_\mathrm{R}$:*

$$\top \sqsubseteq \forall \mathsf{instanceOf}.\top_\mathrm{C} \tag{4.17}$$

$$\top_\mathrm{R} \sqcap \top_\mathrm{C} \sqsubseteq \bot \tag{4.18}$$

$$\begin{aligned} a &: \neg\top_\mathrm{C} \sqcap \neg\top_\mathrm{R} \\ \mathsf{i}_R &: \top_\mathrm{R} \\ \mathsf{i}_A &: \top_\mathrm{C} \end{aligned} \tag{4.19}$$

- URA(N) *consists of axioms* $\{\mathsf{i}_R\} \sqcap \{\mathsf{i}_S\} \sqsubseteq \bot$ *for all pairs of distinct role names* $R, S \in N_\mathrm{R}$.

The following theorem asserts that $\mathcal{K}^1$ is just as strong as $\mathcal{K}$. In the proof, from a model $\mathcal{I}$ of $\mathcal{K}$ we construct a model $\mathcal{J}$ of $\mathcal{K}^1$. Then, we show $\mathcal{J} \models \mathrm{Int}(\varphi)$ iff $\mathcal{I} \models \varphi$. Thus, since $\mathcal{I} \models \mathcal{K}$ and $\mathcal{K}^1 \models \mathrm{Int}(\varphi)$, also $\mathcal{J} \models \mathcal{K}^1$ and hence $\mathcal{J} \models \mathrm{Int}(\varphi)$. Finally, since $\mathcal{J} \models \mathrm{Int}(\varphi)$, also $\mathcal{I} \models \varphi$. The other way around is analogous.

Theorem 2. *For any $\mathcal{HIR}(\mathcal{SROIQ})$ KB $\mathcal{K}$ and any axiom φ in a common vocabulary N, we have $\mathcal{K} \models \varphi$ iff $\mathcal{K}^1 \models \mathrm{Int}(\varphi)$.*

Proof. ($\Leftarrow$): Assume $\mathcal{K}^1 \models \mathrm{Int}(\varphi)$ and take any model $\mathcal{I}$ of $\mathcal{K}$. Let $\mathcal{J} := (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be a first-order interpretation of N^1 where:

$$\begin{aligned} \Delta^\mathcal{J} &:= \Delta^\mathcal{I} & a^\mathcal{J} &:= a^\mathcal{I} \\ \top_\mathrm{C}^\mathcal{J} &:= \Delta_\mathrm{C}^\mathcal{I} & \top_\mathrm{R}^\mathcal{J} &:= \Delta_\mathrm{R}^\mathcal{I} \\ \mathsf{i}_A^\mathcal{J} &:= A^\mathcal{I} & \mathsf{i}_R^\mathcal{J} &:= R^\mathcal{I} \\ A^\mathcal{J} &:= A^{\mathcal{I}\mathcal{E}} & R^\mathcal{J} &:= R^{\mathcal{I}\mathcal{E}} \\ \mathsf{instanceOf}^\mathcal{J} &:= \bigcup_{y \in \Delta_\mathrm{C}^\mathcal{I}} y^\mathcal{E} \times \{y\} \end{aligned}$$

for each $a \in N_\mathrm{I}$, $A \in N_\mathrm{C}$, and $R \in N_\mathrm{R} \setminus \{\mathsf{instanceOf}\}$.

The interpretation $\mathcal{J}$ is easily shown to satisfy InstSync($\mathcal{K}$):

$$\begin{aligned} A^\mathcal{J} = A^{\mathcal{I}\mathcal{E}} &= \{\, x \mid \exists y. x \in y^\mathcal{E} \wedge A^\mathcal{I} = y \,\} \\ &= \{\, x \mid \exists y.(x,y) \in \mathsf{instanceOf}^\mathcal{J} \wedge \mathsf{i}_A^\mathcal{J} = y \,\} \\ &= \{\, x \mid \exists y.(x,y) \in \mathsf{instanceOf}^\mathcal{J} \wedge y \in \{\mathsf{i}_A^\mathcal{J}\} \,\} \\ &= (\exists \mathsf{instanceOf}.\{\mathsf{i}_A\})^\mathcal{J} . \end{aligned}$$

It also satisfies Typing(N): The extension of instanceOf is as a relation in which the second element always belongs to $\Delta_\mathrm{C}^\mathcal{I} = \top_\mathrm{C}^\mathcal{J}$. The disjointness of the extensions of $\top_\mathrm{C}$ and $\top_\mathrm{R}$(4.18) follows from the definition of $\top_\mathrm{C}^\mathcal{J}$, $\top_\mathrm{R}^\mathcal{J}$ and from the disjointness of $\Delta_\mathrm{I}^\mathcal{I}$, $\Delta_\mathrm{C}^\mathcal{I}$ and $\Delta_\mathrm{R}^\mathcal{I}$ (4.2(1)). The classification of each $a \in N_\mathrm{I}$ as well as i_A for each $A \in N_\mathrm{C}$ and i_R for each $R \in N_\mathrm{R}$ (4.19) follows from the definition of $\top_\mathrm{C}^\mathcal{J}$, $\top_\mathrm{R}^\mathcal{J}$, from (4.2(1)), and from (4.2(2)).

URA(N) follows from 4.2(3).

By structural induction we can prove $R^\mathcal{E} = R^\mathcal{J}$ for all role expressions R. Specifically for instanceOf:

$$\begin{aligned}
\mathsf{instanceOf}^{\mathcal{E}} &= \{\, (x,y) \mid x \in \Delta^{\mathcal{I}} \land y \in \Delta^{\mathcal{I}}_{\mathrm{C}} \land x \in y^{\mathcal{E}} \,\} \\
&= \{\, (x,y) \mid y \in \Delta^{\mathcal{I}}_{\mathrm{C}} \land x \in y^{\mathcal{E}} \,\} \\
&= \bigcup_{y \in \Delta^{\mathcal{I}}_{\mathrm{C}}} y^{\mathcal{E}} \times \{y\} \\
&= \mathsf{instanceOf}^{\mathcal{J}} \,.
\end{aligned}$$

By straightforward structral induction, we can also prove $C^{\mathcal{E}} = \mathrm{Int}(C)^{\mathcal{J}}$ for all $\mathcal{HIR}(\mathcal{SROIQ})$-style concepts C. Since nominals containing a role or a concept name are the only case of concept transformed by Int, let's see how it works for them:

$$\{A\}^{\mathcal{E}} = \{A^{\mathcal{I}}\} = \{\mathsf{i}_A^{\mathcal{J}}\} = \{\mathsf{i}_A\}^{\mathcal{J}} = \mathrm{Int}(\{A\})^{\mathcal{J}}$$

$$\{R\}^{\mathcal{E}} = \{R^{\mathcal{I}}\} = \{\mathsf{i}_R^{\mathcal{J}}\} = \{\mathsf{i}_R\}^{\mathcal{J}} = \mathrm{Int}(\{R\})^{\mathcal{J}} \,.$$

This directly implies that $\mathcal{I} \models \varphi$ iff $\mathcal{J} \models \mathrm{Int}(\varphi)$ in case φ is a GCI, RIA or reflexivity or role disjointness assertion. For the remaining cases (concept, role and negative role assertion; $X, Y \in N_{\mathrm{C}} \uplus N_{\mathrm{R}}$):

- $\mathcal{I} \models X : C$ iff $X^{\mathcal{I}} \in C^{\mathcal{E}}$ iff $\mathsf{i}_X^{\mathcal{J}} \in \mathrm{Int}(C)^{\mathcal{J}}$ iff $\mathcal{J} \models \mathrm{Int}(X) : \mathrm{Int}(C)$;
- $\mathcal{I} \models X, Y : R$ iff $(X^{\mathcal{I}}, Y^{\mathcal{I}}) \in R^{\mathcal{E}}$ iff $(\mathsf{i}_X^{\mathcal{J}}, \mathsf{i}_Y^{\mathcal{J}}) \in \mathrm{Int}(R)^{\mathcal{J}}$ iff $\mathcal{J} \models \mathrm{Int}(X), \mathrm{Int}(Y) : \mathrm{Int}(R)$;
- $\mathcal{I} \models X, Y : \neg R$ iff $(X^{\mathcal{I}}, Y^{\mathcal{I}}) \notin R^{\mathcal{E}}$ iff $(\mathsf{i}_X^{\mathcal{J}}, \mathsf{i}_Y^{\mathcal{J}}) \notin \mathrm{Int}(R)^{\mathcal{J}}$ iff $\mathcal{J} \models \mathrm{Int}(X), \mathrm{Int}(Y) : \neg\mathrm{Int}(R)$ iff $\mathcal{J} \models \mathrm{Int}(X), \mathrm{Int}(Y) : \mathrm{Int}(\neg R)$.

From $\mathcal{K}^1 \models \mathrm{Int}(\varphi)$ and $\mathcal{I} \models \mathcal{K}$ now follows $\mathcal{J} \models \mathcal{K}^1$, hence $\mathcal{J} \models \mathrm{Int}(\varphi)$, and finally $\mathcal{I} \models \varphi$.

($\Rightarrow$): Assume $\mathcal{K} \models \varphi$, and take any $\mathcal{J} \models \mathcal{K}^1$. Let $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{E}})$ be an $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation of N where for all $a \in N_{\mathrm{I}}$, $A \in N_{\mathrm{C}}$, $c \in \Delta^{\mathcal{I}}_{\mathrm{C}}$, and $R \in N_{\mathrm{R}} \setminus \{\mathsf{instanceOf}\}$:

$$\begin{aligned}
\Delta^{\mathcal{I}}_{\mathrm{C}} &:= \top^{\mathcal{J}}_{\mathrm{C}} & \Delta^{\mathcal{I}}_{\mathrm{R}} &:= \{\mathsf{i}_R^{\mathcal{J}} \mid R \in N_{\mathrm{R}}\} \\
\Delta^{\mathcal{I}}_{\mathrm{I}} &:= \top^{\mathcal{J}} \setminus (\Delta^{\mathcal{I}}_{\mathrm{C}} \uplus \Delta^{\mathcal{I}}_{\mathrm{R}}) & a^{\mathcal{I}} &:= a^{\mathcal{J}} \\
A^{\mathcal{I}} &:= \mathsf{i}_A^{\mathcal{J}} & R^{\mathcal{I}} &:= \mathsf{i}_R^{\mathcal{J}} \\
c^{\mathcal{E}} &:= \{\, x \mid (x,c) \in \mathsf{instanceOf}^{\mathcal{J}} \,\} & R^{\mathcal{I}\mathcal{E}} &:= R^{\mathcal{J}} \,.
\end{aligned}$$

Note that thanks to the definition of $\Delta^{\mathcal{I}}_{\mathrm{R}}$ by defining $R^{\mathcal{I}\mathcal{E}}$ for each $R \in N_{\mathrm{R}}$, we define $r^{\mathcal{E}}$ for each $r \in \Delta^{\mathcal{I}}_{\mathrm{R}}$.

The requirements for an interpretation 4.2(1) and 4.2(4) follow from the definition of $\mathcal{I}$ and Typing(N) (4.18). The requirement 4.2(2) follows from Typing(N) (4.19) and the definitions of $\Delta^{\mathcal{I}}_{\mathrm{C}}$, $\Delta^{\mathcal{I}}_{\mathrm{R}}$, and $\Delta^{\mathcal{I}}_{\mathrm{I}}$. The requirement 4.2(3) follows from URA(N).

By structural induction, we can prove $S^{\mathcal{E}} = S^{\mathcal{J}}$ for all $\mathcal{HIR}(\mathcal{SROIQ})$-style role expressions S. Let us show it specifically for $\mathsf{instanceOf}$. First, observe that since $\mathcal{J}$ satisfies Typing(N), $\{\, y \mid \exists x.(x,y) \in \mathsf{instanceOf}^{\mathcal{J}} \,\} \subseteq \top^{\mathcal{J}}_{\mathrm{C}}$. Now, for $\mathsf{instanceOf}$:

$$\begin{aligned}
\mathsf{instanceOf}^{\mathcal{E}} &= \{\, (x,y) \mid x \in \Delta^{\mathcal{I}} \wedge y \in \Delta^{\mathcal{I}}_{\mathrm{C}} \wedge x \in y^{\mathcal{E}} \,\} \\
&= \{\, (x,y) \mid x \in y^{\mathcal{E}} \,\} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}_{\mathrm{C}} \\
&= \mathsf{instanceOf}^{\mathcal{J}} \cap \top^{\mathcal{J}} \times \top^{\mathcal{J}}_{\mathrm{C}} \\
&= \mathsf{instanceOf}^{\mathcal{J}} .
\end{aligned}$$

Since $\mathcal{J}$ satisfies InstSync, $A^{\mathcal{J}} = \{\, x \mid (x, \mathsf{i}_A) \in \mathsf{instanceOf}^{\mathcal{J}} \,\} = \mathsf{i}^{\mathcal{E}}_A = A^{\mathcal{IE}}$. Analogously to the proof of ($\Leftarrow$) we can show (by structural induction) also that $\mathrm{Int}(D)^{\mathcal{J}} = D^{\mathcal{E}}$ for all $\mathcal{HIR}(\mathcal{SROIQ})$-style concepts D.

This implies that $\mathcal{I} \models \varphi$ iff $\mathcal{J} \models \mathrm{Int}(\varphi)$ in case φ is a GCI, RIA or reflexivity or role disjointness assertion. For the remaining cases, the equivalence can be proved just as in the ($\Leftarrow$) part of the proof. From $\mathcal{K} \models \varphi$ and $\mathcal{J} \models \mathcal{K}^1$ now follows $\mathcal{I} \models \mathcal{K}$, hence $\mathcal{I} \models \varphi$, and finally $\mathcal{J} \models \mathrm{Int}(\varphi)$. □

Observe that the size of $\mathcal{K}^1$ (string-length-wise) is at most quadratic in the size of $\mathcal{K}$ (assuming N consists exactly of all symbols occurring in $\mathcal{K}$). If $\mathcal{K}$ satisfies, for all roles including $\mathsf{instanceOf}$, the $\mathcal{SROIQ}$ restrictions as mentioned at the end of Subsection 1.1.3, so does $\mathcal{K}^1$. With simple roles defined as for $\mathcal{SROIQ}$, we can now state the following corollary claiming that $\mathcal{HIR}(\mathcal{SROIQ})$ has the same complexity as first-order $\mathcal{SROIQ}$.

Corollary 1. *Let a $\mathcal{HIR}(\mathcal{SROIQ})$ KB $\mathcal{K}$ be such that only simple roles occur in cardinality restrictions. Concept satisfiability and entailment in a $\mathcal{HIR}(\mathcal{SROIQ})$ KB are then decidable in* $\mathsf{N2ExpTime}$.

From Definition 4.4 it follows that in general, $\mathcal{HIR}(\mathcal{L})$ reduces to $\mathcal{LO}$ if the DL $\mathcal{L}$ admits GCIs, existential restriction, and complement. The decidability and complexity of standard reasoning tasks for $\mathcal{HIR}(\mathcal{L})$ are then the same as for $\mathcal{LO}$.

4.4 Subsumption Metamodelling with $\mathcal{HIRS}_*(\mathcal{L})$ Logics

As already mentioned in Chapter 2, a natural next step after metamodelling instantiation is to seek to obtain a similar role $\mathsf{subClassOf}$ reflecting subsumption. Such a role has more possible uses. The one we consider the most useful allows to select subconcepts of a given concept satisfying various criteria, e.g., to express that all species of genus Giraffa except Giraffa camelopardalis are extinct:

$$\mathsf{Species} \sqcap \exists \mathsf{subClassOf}.\{\mathsf{Giraffa}\} \sqcap \neg\{\mathsf{Giraffa\ camelopardalis}\} \sqsubseteq \mathsf{Extinct}. \tag{4.20}$$

More generally, the subClassOf role can express subsumption relationships among instances of two meta-concepts. For instance, in the biological taxonomy, species are grouped into higher-ranked taxa. While some ranks are only used in some branches of biology, certain ranks are generally anticipated as required, and each species should have a supertaxon of those ranks (4.21). We might even wish to express that these relationships are functional by axioms such as (4.22) (similar to the example use (2.9) from Chapter 2).

$$\begin{gathered}\mathsf{Species} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Genus}\\ \mathsf{Genus} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Family}\\ \mathsf{Family} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Order}\\ \mathsf{Order} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Kingdom}\end{gathered} \tag{4.21}$$

$$\mathsf{Species} \sqsubseteq \leqslant 1\,\mathsf{subClassOf}.\mathsf{Genus} \tag{4.22}$$

Since subClassOf is expected to be transitive, the meta subsumptions (4.21) should imply that each Species is a subclass of some Order (4.23) even though we may not know the precise Genus and Family lying in-between. Moreover, instantiation is expected to propagate upward the subsumption chain. If zarafa is an instance of Giraffa camelopardalis, which is a Species (4.24), we expect that axioms (4.21) imply that it is also an instance of some Kingdom (4.25).

$$\mathsf{Species} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Order} \tag{4.23}$$

$$\begin{gathered}\mathsf{zarafa} : \mathsf{Giraffa\ camelopardalis}\\ \mathsf{Giraffa\ camelopardalis} : \mathsf{Species}\end{gathered} \tag{4.24}$$

$$\mathsf{zarafa} : \exists \mathsf{instanceOf}.\mathsf{Kingdom} \tag{4.25}$$

The zoological taxonomy (consisting of descendant taxa of the Animalia kingdom) has two more ranks of taxa above Order: Class and Phylum. We can assert their existence in the hierarchy by somewhat more complex axioms:

$$\begin{gathered}\mathsf{Order} \sqcap \exists \mathsf{subClassOf}.\{\mathsf{Animalia}\} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Class}\\ \mathsf{Class} \sqcap \exists \mathsf{subClassOf}.\{\mathsf{Animalia}\} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Phylum}\\ \mathsf{Phylum} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Kingdom}.\end{gathered} \tag{4.26}$$

In the above axioms as well as in (4.20), concepts are selected on the basis of being subsumed by a concept (Animalia and Giraffa, respectively). This brings forward the question of semantics of the subClassOf role. Since this is a meta-level role between two intensions, its semantics need not be strictly based on their extensions.

There is little doubt that the necessary condition of subsumption,

$$\forall c \forall d[\mathsf{subClassOf}(c,d) \Rightarrow \forall x(\mathsf{instanceOf}(x,c) \Rightarrow \mathsf{instanceOf}(x,d))]\,, \tag{4.27}$$

should hold, i.e., the meta-level assertion of subClassOf between two concepts implies the respective subset relationship of their extensions.

The sufficient condition of subsumption converse to (4.27),

$$\forall c \forall d[\forall x(\mathsf{instanceOf}(x,c) \Rightarrow \mathsf{instanceOf}(x,d)) \Rightarrow \mathsf{subClassOf}(c,d)]\,, \tag{4.28}$$

is more controversial from the philosophical point of view (Guizzardi, 2005). Description logics strongly support the nominalist approach to philosophical ontology. Nominalism understands universals (i.e., entities which can have instances) as predicates that can be stated over particulars (i.e., individual entities which cannot have instances). The ontological subtype relation thus coincides with the set-theoretical subset relation on extensions of types. In order to support this approach, the subClassOf role should also satisfy (4.28). An illustration of the consequences of this property is that if every instance of any species is an organism, $\exists\mathsf{instanceOf}.\mathsf{Species} \sqsubseteq \mathsf{Organism}$, then an intuitive consequence is entailed – each species is a subconcept of the concept Organism: $\mathsf{Species} \sqsubseteq \exists\mathsf{subClassOf}.\{\mathsf{Organism}\}$.

Set-theoretical subsumption, however, has also counterintuitive consequences, such as empty types (e.g., Unicorn) being subtypes of all types. Guizzardi (2005) thus argues for determining whether one ontological type is a subtype of another by comparing their intended meaning, which is based on common properties of their respective instances in possible worlds. Guizzardi's notion of intended meaning is similar to the definition of intension by Carnap (1947): an intension denotes the "property" or "character" of a name. Conversely, an extension denotes the "class" corresponding to the name. (See also Section 5.2.) From this perspective, the subtype relationship should be based on intensions, which stay the same in any state of the world, and not extensions, which can vary in different states of the world.

It is out of the scope of this thesis to discuss whether such a notion of subtype can be expressed in a description logic. Nevertheless, we should be aware that a metamodel of subsumption satisfying the sufficient condition (4.28) produces consequences undesirable in some applications. We will therefore investigate also weaker, non-set-theoretical metamodels, satisfying only reflexivity, transitivity, and the necessary condition (4.27).

Such metamodels can be extended in the future with more sophisticated notions of subtypes based on the structure of intensions, which is opaque in our current semantics.

In the following subsections we discuss three options for metamodelling the subsumption relation, for which we are able to obtain decidability via reduction as in the case of $\mathcal{HIR}(\mathcal{L})$. As we already mentioned in the introduction to this chapter, together we denote them $\mathcal{HIRS}_*(\mathcal{L})$. They all use the same syntax:

Definition 4.5 ($\mathcal{HIRS}_*(\mathcal{SROIQ})$ Syntax)**.** *An* $\mathcal{HIRS}_*(\mathcal{SROIQ})$ vocabulary *is a DL vocabulary* $N = N_\mathrm{C} \uplus N_\mathrm{R} \uplus N_\mathrm{I}$ *such that* $\{\mathsf{instanceOf}, \mathsf{subClassOf}\} \in N_\mathrm{R}$.

$\mathcal{HIRS}_*(\mathcal{SROIQ})$ role expressions*,* concepts*,* axioms*, and* knowledge base *in* N *are defined identically to their respective* $\mathcal{HIR}(\mathcal{SROIQ})$ *counterparts.*

First two logics $\mathcal{HIRS}_\mathrm{NN}(\mathcal{L})$ and $\mathcal{HIRS}_\mathrm{NA}(\mathcal{L})$ are based on the non-set-theoretical notion, while the last, $\mathcal{HIRS}_\mathrm{SN}(\mathcal{L})$ on the set-theoretical notion of subsumption. As already mentioned in the introduction to this chapter, together we denote these logics as $\mathcal{HIRS}_*(\mathcal{L})$. We define each of these logics as an extension of $\mathcal{HIR}(\mathcal{SROIQ})$, although they can be easily modified to extend $\mathcal{HIR}(\mathcal{L})$.

4.4.1 Non-Set-Theoretical Subsumption for Named Concepts

Let us first introduce the extension $\mathcal{HIRS}_\mathrm{NN}(\mathcal{SROIQ})$ of $\mathcal{HIR}(\mathcal{SROIQ})$ with non-set-theoretical metamodel of subsumption whose properties are guaranteed only on named concepts. (Hence the name $\mathcal{HIRS}_\mathrm{NN}(\mathcal{L})$, where the first N stands for "non-set-theoretical" and the second N for "named concepts".) While rather weak compared to the other options, this approach has the advantage that the use of the roles instanceOf and subClassOf in modelling is not restricted more than in $\mathcal{SROIQ}$.

Definition 4.6 ($\mathcal{HIRS}_\mathrm{NN}(\mathcal{SROIQ})$ Semantics)**.** *An* $\mathcal{HIRS}_\mathrm{NN}(\mathcal{SROIQ})$ interpretation *of an* $\mathcal{HIRS}(\mathcal{SROIQ})$ *vocabulary* N *is a* $\mathcal{HIR}(\mathcal{SROIQ})$ *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, \cdot^\mathcal{E})$ *where additionally* $\mathsf{subClassOf}^{\mathcal{IE}} \subseteq \Delta^\mathcal{I}_\mathrm{C} \times \Delta^\mathcal{I}_\mathrm{C}$*, and for all* c*,* $d \in \Delta^\mathcal{I}_\mathrm{C}$*,* $A \in N_\mathrm{C}$*:*

1. $(A^\mathcal{I}, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}}$,
2. *if* $(c, d), (d, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}}$*, then* $(c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}}$,
3. *if* $(c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}}$*, then* $c^\mathcal{E} \subseteq A^{\mathcal{IE}}$.

The extension of $\mathcal{HIRS}_\mathrm{NN}(\mathcal{SROIQ})$ *interpretation to* $\mathcal{HIRS}(\mathcal{SROIQ})$ *role expressions and concepts, satisfaction of axioms, model, satisfiability, etc. are defined analogously to* $\mathcal{HIR}(\mathcal{SROIQ})$.

The following proposition shows that the description logic $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ (i.e., the $\mathcal{HIRS}(\mathcal{SROIQ})$ syntax under the $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ semantics) supports non-set-theoretic semantics of $\mathsf{subClassOf}$ on named (i.e., atomic) concepts. Definition 4.6 is slightly more general for the sake of a simple and efficient reduction, which we provide below.

Proposition 1. *Let $\mathcal{I}$ be a $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ interpretation of a vocabulary N. Then the extension $\mathsf{subClassOf}^{\mathcal{IE}}$ is a reflexive and transitive relation on the set $N_{\mathrm{C}}^{\mathcal{I}}$ of intensions of all atomic concepts, and for all A, $B \in N_{\mathrm{C}}$, if $(A^{\mathcal{I}}, B^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$, then $A^{\mathcal{IE}} \subseteq B^{\mathcal{IE}}$.*

Proof. The reflexivity of $\mathsf{subClassOf}^{\mathcal{IE}}$ is obvious from 4.6(1). The transitivity follows from 4.6(2) and the fact that $N_{\mathrm{C}}^{\mathcal{I}} \subseteq \Delta_{\mathrm{C}}^{\mathcal{I}}$. Similarly, the last part (if $(A^{\mathcal{I}}, B^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$, then $A^{\mathcal{IE}} \subseteq B^{\mathcal{IE}}$) follows from 4.6(3) and $N_{\mathrm{C}}^{\mathcal{I}} \subseteq \Delta_{\mathrm{C}}^{\mathcal{I}}$. □

As mentioned above, the use of both $\mathsf{instanceOf}$ and $\mathsf{subClassOf}$ in $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ KBs is not restricted more than in $\mathcal{SROIQ}$. They can be used in number restrictions to model, e.g., disjointness of concepts (4.14) or functional subsumption relationships among selected concepts (4.22), provided they remain simple in the respective KB. Or, they can be used in RIAs as any other role and possibly become non-simple.

The $\mathsf{subClassOf}$ role in $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ does not, of course, satisfy the sufficient condition for subsumption (4.28) for named or unnamed concepts. It also does not metamodel subsumption of unnamed concepts properly. For them, $\mathsf{subClassOf}$ does not satisfy transitivity (axioms (4.21) do not entail (4.23)), nor the necessary condition (4.27) ((4.21) together with (4.24) do not entail (4.25)). However, the metamodel has information on unnamed subconcepts of named superconcepts and propagates instances of unnamed subconcepts to their named superconcepts. Thus, if we add to the hierarchy (4.21) the claim that every $\mathsf{Kingdom}$ is subsumed by the concept $\mathsf{Organism}$:

$$\mathsf{Kingdom} \sqsubseteq \exists \mathsf{subClassOf}.\{\mathsf{Organism}\}, \tag{4.29}$$

axioms (4.24) now entail $\mathsf{zarafa}\colon \mathsf{Organism}$, but still not (4.25).

$\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ can be reduced to $\mathcal{SROIQ}$ by extending the $\mathcal{HIR}(\mathcal{SROIQ})$ reduction with an axiomatization of the $\mathsf{subClassOf}$ role. The size of this axiomatization is linear in the number of atomic concepts in the vocabulary.

Definition 4.7 ($\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ First-Order Reduction)**.** *Let $\mathcal{K}$ be a $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ KB in a vocabulary $N = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$. The $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$* first-order reduction of *$\mathcal{K}$ is the $\mathcal{SROIQ}$ KB $\mathcal{K}^{1\mathrm{NN}} := \mathcal{K}^1 \cup \mathrm{SubclassSync}(\mathcal{K})$, where* *the KB $\mathcal{K}^1$ in the vocabulary N^1 is the $\mathcal{HIR}(\mathcal{SROIQ})$ first-order reduction of $\mathcal{K}$ (cf. Def. 4.4), and* SubclassSync($\mathcal{K}$) *consists of the following axioms for every $A \in N_\mathrm{C}$:*

$$\begin{aligned} \exists\mathsf{subClassOf}.\top &\sqsubseteq \top_\mathrm{C} \\ \top &\sqsubseteq \forall\mathsf{subClassOf}.\top_\mathrm{C} \end{aligned} \tag{4.30}$$

$$\mathsf{i}_A, \mathsf{i}_A : \mathsf{subClassOf} \tag{4.31}$$

$$\exists\mathsf{subClassOf}.\exists\mathsf{subClassOf}.\{\mathsf{i}_A\} \sqsubseteq \exists\mathsf{subClassOf}.\{\mathsf{i}_A\} \tag{4.32}$$

$$\exists\mathsf{instanceOf}.\exists\mathsf{subClassOf}.\{\mathsf{i}_A\} \sqsubseteq A. \tag{4.33}$$

The following theorem asserts that $\mathcal{K}^{1\mathrm{NN}}$ is just as strong as $\mathcal{K}$.

Theorem 3. *For any $\mathcal{HIRS}_\mathrm{NN}(\mathcal{SROIQ})$ KB $\mathcal{K}$ and any axiom φ in a common vocabulary N, we have $\mathcal{K} \models \varphi$ iff $\mathcal{K}^{1\mathrm{NN}} \models \mathrm{Int}(\varphi)$.*

Proof. ($\Leftarrow$): Assume $\mathcal{K}^{1\mathrm{NN}} \models \mathrm{Int}(\varphi)$ and take any model $\mathcal{I}$ of $\mathcal{K}$. Let $\mathcal{J} := (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be a first-order interpretation of N^1 defined exactly as in the ($\Leftarrow$) direction of the proof of Theorem 2. From the proof of Thm. 2 follows that $\mathcal{J}$ satisfies Int($\mathcal{K}$), InstSync($\mathcal{K}$) and Typing($\mathcal{K}$).

Let us show that $\mathcal{J}$ satisfies also SubclassSync($\mathcal{K}$). Clearly, axioms (4.30) are satisfied due to the definition of $\top_\mathrm{C}^\mathcal{J}$ and the fact that $\mathsf{subClassOf}^\mathcal{J} = \mathsf{subClassOf}^{\mathcal{IE}} \subseteq \Delta_\mathrm{C}^\mathcal{I} \times \Delta_\mathrm{C}^\mathcal{I}$. Axioms (4.31) are satisfied due to Def. 4.6(1). As for axioms (4.32), for any given $A \in N_\mathrm{C}$ we have:

$$\begin{aligned} &(\exists\mathsf{subClassOf}.\exists\mathsf{subClassOf}.\{\mathsf{i}_A\})^\mathcal{J} \\ &\quad = \{\, c \mid \exists d(d \in \top_\mathrm{C}^\mathcal{J} \wedge (c,d) \in \mathsf{subClassOf}^\mathcal{J} \wedge (d, \mathsf{i}_A^\mathcal{J}) \in \mathsf{subClassOf}^\mathcal{J})\,\} \\ &\quad = \{\, c \mid \exists d(d \in \Delta_\mathrm{C}^\mathcal{I} \wedge (c,d) \in \mathsf{subClassOf}^{\mathcal{IE}} \wedge (d, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}})\,\} \\ &\quad \overset{(*)}{\subseteq} \{\, c \mid (c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}} \,\} \\ &\quad = \{\, c \mid (c, \mathsf{i}_A^\mathcal{J}) \in \mathsf{subClassOf}^\mathcal{J} \,\} \\ &\quad = (\exists\mathsf{subClassOf}.\{\mathsf{i}_A\})^\mathcal{J}, \end{aligned}$$

where the inclusion $(*)$ holds by Def. 4.6(2).

As for axioms (4.33), for any given $A \in N_\mathrm{C}$, we have

$$\begin{aligned} &(\exists\mathsf{instanceOf}.\exists\mathsf{subClassOf}.\{\mathsf{i}_A\})^\mathcal{J} \\ &\quad = \{\, b \mid \exists c(c \in \top_\mathrm{C}^\mathcal{J} \wedge (b,c) \in \mathsf{instanceOf}^\mathcal{J} \wedge (c, \mathsf{i}_A^\mathcal{J}) \in \mathsf{subClassOf}^\mathcal{J})\,\} \\ &\quad = \{\, b \mid \exists c(c \in \Delta_\mathrm{C}^\mathcal{I} \wedge (b,c) \in \mathsf{instanceOf}^{\mathcal{IE}} \wedge (c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}})\,\} \\ &\quad \overset{(*)}{=} \{\, b \mid \exists c(c \in \Delta_\mathrm{C}^\mathcal{I} \wedge (b,c) \in \mathsf{instanceOf}^{\mathcal{IE}} \wedge c \subseteq A^\mathcal{I})\,\} \\ &\quad \subseteq \{\, b \mid (b, A^\mathcal{I}) \in \mathsf{instanceOf}^{\mathcal{IE}} \,\} \\ &\quad = \{\, b \mid b \in A^{\mathcal{IE}} \,\} = A, \end{aligned}$$

where the inclusion $(*)$ holds by Def. 4.6(3). The rest of the proof for this direction is analogous to the ($\Leftarrow$) direction of the proof of Thm. 2.

($\Rightarrow$): Assume $\mathcal{K} \models \varphi$, and take any $\mathcal{J} \models \mathcal{K}^{1\text{NN}}$. Let $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{E}})$ be an $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation of N defined exactly as in the ($\Rightarrow$) direction of the proof of Thm. 2. We know from the proof of Thm. 2 that $R^{\mathcal{E}} = R^{\mathcal{J}}$ for all role expressions R, including $\mathsf{instanceOf}$ and $\mathsf{subClassOf}$. We will show that $\mathcal{I}$ is also a $\mathcal{HIRS}_{\text{NN}}(\mathcal{SROIQ})$ interpretation by verifying conditions from Def. 4.6.

The first condition holds due to the definition of $A^{\mathcal{I}}$ and (4.31). As for the second condition, take any $A \in N_{\text{C}}$, c, $d \in \Delta^{\mathcal{I}}_{\text{C}}$, and assume (c, d), $(d, A^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$. Then also (c, d), $(d, \mathsf{i}^{\mathcal{J}}_A) \in \mathsf{subClassOf}^{\mathcal{J}}$ by the definition of $A^{\mathcal{I}}$ and the fact that $\mathsf{subClassOf}^{\mathcal{IE}} = \mathsf{subClassOf}^{\mathcal{J}}$, hence $(c, \mathsf{i}^{\mathcal{J}}_A) \in \mathsf{subClassOf}^{\mathcal{J}}$ by axiom (4.32). Therefore $(c, A^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$ by the definition of $A^{\mathcal{I}}$ again.

Similarly for the third condition: take any $A \in N_{\text{C}}$, $c \in \Delta^{\mathcal{I}}_{\text{C}}$, and assume $(c, A^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$. Then also $(c, \mathsf{i}^{\mathcal{J}}_A) \in \mathsf{subClassOf}^{\mathcal{J}}$ by definition of $A^{\mathcal{I}}$ and the fact that $R^{\mathcal{IE}} = R^{\mathcal{J}}$ for all $R \in N_{\text{R}}$. We will show that if $x \in c^{\mathcal{E}}$, then $x \in A^{\mathcal{IE}}$. Since $x \in c^{\mathcal{E}}$, also $(x, c) \in \mathsf{instanceOf}^{\mathcal{IE}}$ by the definition of $c^{\mathcal{E}}$ and thus $(x, c) \in \mathsf{instanceOf}^{\mathcal{J}}$. Further, from (4.33) and $(x, c) \in \mathsf{instanceOf}^{\mathcal{J}}$ with $(c, \mathsf{i}^{\mathcal{J}}_A) \in \mathsf{subClassOf}^{\mathcal{J}}$ we get $x \in A^{\mathcal{J}}$. Finally, from $A^{\mathcal{J}} = A^{\mathcal{IE}}$ (proved in the ($\Rightarrow$) direction of the proof of Thm. 2) follows $x \in A^{\mathcal{IE}}$.

The rest of the proof is analogous to the ($\Rightarrow$) direction of the proof of the Thm. 2. □

Note that we could replace Def. 4.6(1) and (4.31) with $(c, c) \in \mathsf{subClassOf}^{\mathcal{IE}}$ and $\top_{\text{C}} \sqsubseteq \exists \mathsf{subClassOf}.\mathsf{Self}$, respectively. However, to ensure that $\mathcal{HIRS}_{\text{NN}}(\mathcal{SROIQ})$ KB is decidable, $\mathsf{subClassOf}$ would have to be a simple role.

4.4.2 Non-Set-Theoretical Subsumption for All Concepts

The second semantics of $\mathsf{subClassOf}$ that we present is a non-set-theoretical one whose properties are the same for named and unnamed concepts. However, for the sake of decidability, $\mathsf{instanceOf}$ and $\mathsf{subClassOf}$ are considered non-simple. The logic is called $\mathcal{HIRS}_{\text{NA}}(\mathcal{L})$ (N for "non-set-theoretical" and A for "all concepts").

Definition 4.8 ($\mathcal{HIRS}_{\text{NA}}(\mathcal{SROIQ})$ Semantics)**.** *An $\mathcal{HIRS}_{\text{NA}}(\mathcal{SROIQ})$ interpretation of a $\mathcal{HIRS}(\mathcal{SROIQ})$ vocabulary N is a $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{E}})$ where additionally*

1. $\mathsf{subClassOf}^{\mathcal{IE}}$ *is a transitive relation on $\Delta^{\mathcal{I}}_{\text{C}}$, and*
2. *for all c, $d \in \Delta^{\mathcal{I}}_{\text{C}}$: if $(c, d) \in \mathsf{subClassOf}^{\mathcal{IE}}$, then $c^{\mathcal{E}} \subseteq d^{\mathcal{E}}$.*

The extension of $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ interpretation to $\mathcal{HIRS}(\mathcal{SROIQ})$ role expressions and concepts, satisfaction of axioms, model, satisfiability, etc. are defined analogously to $\mathcal{HIR}(\mathcal{SROIQ})$.

Under this semantics the meta model of subsumption satisfies the necessary condition (4.27). Transitivity is also satisfied, and so (4.21) (axioms making each species a subclass of some genus, each genus a subclass of some family, each family a subclass of some order and each order a subclass of some kingdom) entails (4.23) (axiom asserting that each species is a subclass of some order). Further, (4.21) together with (4.24) (axioms classifying zarafa as an individual of species Giraffa camelopardalis and Giraffa camelopardalis as a Species) entail (4.25) (zarafa is an instance of some kingdom). The sufficient condition for subsumption (4.28) is not enforced.

$\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ KBs can be reduced to first-order $\mathcal{SROIQ}$ KBs as follows:

Definition 4.9 ($\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ First-Order Reduction)**.** *Let $\mathcal{K}$ be a $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ KB in a vocabulary $N = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$. The $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$* first-order reduction $\mathcal{K}^{1\mathrm{NA}}$ of $\mathcal{K}$ *is the $\mathcal{SROIQ}$ KB $\mathcal{K}^{1\mathrm{NA}} := \mathcal{K}^1 \cup \mathrm{SubclassSync}(\mathcal{K})$, where the $\mathcal{SROIQ}$ KB $\mathcal{K}^1$ in the vocabulary N^1 is the first-order reduction of $\mathcal{K}$ as defined in Def. 4.4, and* $\mathrm{SubclassSync}(\mathcal{K})$ *consists of the following axioms:*

$$\begin{aligned} \exists \mathsf{subClassOf}.\top &\sqsubseteq \top_{\mathrm{C}} \\ \top &\sqsubseteq \forall \mathsf{subClassOf}.\top_{\mathrm{C}} \end{aligned} \tag{4.34}$$

$$\mathsf{subClassOf} \cdot \mathsf{subClassOf} \sqsubseteq \mathsf{subClassOf} \tag{4.35}$$

$$\mathsf{instanceOf} \cdot \mathsf{subClassOf} \sqsubseteq \mathsf{instanceOf}. \tag{4.36}$$

Theorem 4. *For any $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ KB $\mathcal{K}$ and any axiom φ in a common vocabulary N, we have $\mathcal{K} \models \varphi$ iff $\mathcal{K}^{1\mathrm{NA}} \models \mathrm{Int}(\varphi)$.*

If $\mathcal{K}$ statisfies the $\mathcal{SROIQ}$ requirements for decidability with instanceOf *and* subClassOf *considered non-simple and the regular order of roles such that* subClassOf $\prec$ instanceOf*, then concept satisfiability and entailment in $\mathcal{K}$ are decidable.*

Proof. ($\Leftarrow$): Assume $\mathcal{K}^{1\mathrm{NA}} \models \mathrm{Int}(\varphi)$ and take any model $\mathcal{I}$ of $\mathcal{K}$. Let $\mathcal{J} := (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be a first-order interpretation of N^1 defined exactly as in the ($\Leftarrow$) direction of the proof of Theorem 2. From the proof of Thm. 2 follows that $\mathcal{J}$ satisfies $\mathrm{Int}(\mathcal{K})$, $\mathrm{InstSync}(\mathcal{K})$ and $\mathrm{Typing}(\mathcal{K})$.

Let us show that $\mathcal{J}$ satisfies also $\mathrm{SubclassSync}(\mathcal{K})$. Clearly, axioms (4.34) are satisfied due to the definition of $\top_{\mathrm{C}}^{\mathcal{J}}$ and the fact that subClassOf is a relation on $\Delta_{\mathrm{C}}^{\mathcal{I}}$. Axiom (4.35) is satisfied due to the transitivity of $\mathsf{subClassOf}^{\mathcal{J}}$ following from Definition 4.8(1). As for the last axiom, (4.36), take any $a \in \Delta^{\mathcal{I}}$, $b, c \in \top_{\mathrm{C}}^{\mathcal{J}}$. Let (a, b) $\in \mathsf{instanceOf}^{\mathcal{J}}$ and $(b, c) \in \mathsf{subClassOf}^{\mathcal{J}}$. From $R^{\mathcal{J}} = R^{\mathcal{I}\mathcal{E}}$ for each $R \in N_{\mathrm{R}}$

(proved in the ($\Leftarrow$) direction of the proof of Thm. 2) we get $(a,b) \in \mathsf{instanceOf}^{\mathcal{IE}}$ and $(b,c) \in \mathsf{subClassOf}^{\mathcal{IE}}$. Then $a \in b^{\mathcal{E}}$ (from the definition of $\mathsf{instanceOf}^{\mathcal{IE}}$) and $b^{\mathcal{E}} \subseteq c^{\mathcal{E}}$ (from 4.8(2)). Thus $a \in c^{\mathcal{E}}$, and from the definition of $\mathsf{instanceOf}^{\mathcal{IE}}$ also $(a,c) \in \mathsf{instanceOf}^{\mathcal{IE}}$. Finally, $(a,c) \in \mathsf{instanceOf}^{\mathcal{J}}$.

The rest of the proof for this direction is analogous to the ($\Leftarrow$) direction of the proof of Thm. 2.

($\Rightarrow$): Assume $\mathcal{K} \models \varphi$, and take any $\mathcal{J} \models \mathcal{K}^{1\mathrm{NA}}$. Let $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{E}})$ be an $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation of N defined exactly as in the ($\Rightarrow$) direction of the proof of Thm. 2. We will show that $\mathcal{I}$ is also a $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ interpretation by verifying conditions from Def. 4.8.

The first condition (4.8(1)) holds due to the axioms (4.34), (4.35) and the definition of $\Delta^{\mathcal{I}}_{\mathrm{C}}$. As for the second condition (4.8(2)), take any $c,d \in \Delta^{\mathcal{I}}_{\mathrm{C}}$ and assume $(c,d) \in \mathsf{subClassOf}^{\mathcal{IE}}$. Then also $(c,d) \in \mathsf{subClassOf}^{\mathcal{J}}$ since $R^{\mathcal{IE}} = R^{\mathcal{J}}$ for each $R \in N_{\mathrm{R}}$ (proved in the ($\Rightarrow$) direction of the proof of Thm. 2). We will show that if $x \in c^{\mathcal{E}}$, then $x \in d^{\mathcal{E}}$. Since $x \in c^{\mathcal{E}}$, also $(x,c) \in \mathsf{instanceOf}^{\mathcal{IE}}$ by the definition of $c^{\mathcal{E}}$ and thus $(x,c) \in \mathsf{instanceOf}^{\mathcal{J}}$. Further, from (4.36) and $(x,c) \in \mathsf{instanceOf}^{\mathcal{J}}$ with $(c,d) \in \mathsf{subClassOf}^{\mathcal{J}}$ we get $(x,d) \in \mathsf{instanceOf}^{\mathcal{J}}$. Finally, $(x,d) \in \mathsf{instanceOf}^{\mathcal{IE}}$ and from the definition of $d^{\mathcal{E}}$ thus also $x \in d^{\mathcal{E}}$. The rest of the proof is analogous to the ($\Rightarrow$) direction of the proof of the Thm. 2. □

Unlike in the cases of $\mathcal{HIR}(\mathcal{SROIQ})$ and $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$, the decidability of $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{SROIQ})$ requires posing additional constraints on its expressivity. Since $\mathcal{SROIQ}$ does not admit Self restriction on non-simple roles, reflexivity of subClassOf cannot be axiomatized. This has no effect on propagation of instances from sub- to superconcepts, and can be compensated for in descriptions by replacing $\exists\mathsf{subClassOf}.C$ with $C \sqcup \exists\mathsf{subClassOf}.C$, and $\forall\mathsf{subClassOf}.C$ with $C \sqcap \forall\mathsf{subClassOf}.C$. The lack of reflexivity is thus rather negligible, directly affecting only RIAs.

A more inconvenient limitation is the prohibition of number restrictions on both instanceOf and subClassOf, which prevents, e.g., the useful axiomatizations of meta concepts of disjoint concepts (4.14), as well as assertions of functionality of subClassOf (4.22).

In addition, note that the reduced knowledge base $\mathcal{K}^{1\mathrm{NA}}$ for any $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{L})$ knowledge base $\mathcal{K}$ contains a transitivity assertion and a complex role inclusion axiom. Thus, for $\mathcal{HIRS}_{\mathrm{NA}}(\mathcal{L})$ to be reduced to $\mathcal{LO}$, $\mathcal{L}$ has to admit not only GCIs, existential restriction and complement, as in case of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{L})$, but also complex RIAs.

4.4.3 Set-Theoretical Subsumption for Named Concepts

We conclude our exploration of subsumption metamodelling options with the logic $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ featuring set-theoretical metamodel of subsumption on named concepts. Similarly to $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$, the restriction of the subsumption metamodel to named concepts allows more liberal use of $\mathsf{instanceOf}$ and $\mathsf{subClassOf}$ in KBs.

Definition 4.10 ($\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ Semantics)**.** *An $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ interpretation of a $\mathcal{HIRS}(\mathcal{SROIQ})$ vocabulary N is a $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{E}})$ where additionally*

1. $\mathsf{subClassOf}^{\mathcal{IE}} \subseteq \Delta^{\mathcal{I}}_{\mathrm{C}} \times \Delta^{\mathcal{I}}_{\mathrm{C}}$*, and*
2. $(c, A^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$ *iff* $c^{\mathcal{E}} \subseteq A^{\mathcal{IE}}$ *for all* $A \in N_{\mathrm{C}}$ *and* $c \in \Delta^{\mathcal{I}}_{\mathrm{C}}$.

The extension of $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ interpretation to $\mathcal{HIRS}(\mathcal{SROIQ})$ role expressions and concepts, satisfaction of axioms, model, satisfiability, etc. are defined analogously to $\mathcal{HIR}(\mathcal{SROIQ})$.

The following proposition states that $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ indeed fully metamodels set-theoretical subsumption on named concepts.

Proposition 2. *Let $\mathcal{I}$ be a $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ interpretation of a vocabulary N. Then for all A, $B \in N_{\mathrm{C}}$, we have $(A^{\mathcal{I}}, B^{\mathcal{I}}) \in \mathsf{subClassOf}^{\mathcal{IE}}$ iff $A^{\mathcal{IE}} \subseteq B^{\mathcal{IE}}$.*

Proof. The proposition directly follows from 4.10(2) and the fact that $N^{\mathcal{I}}_{\mathrm{C}} \subseteq \Delta^{\mathcal{I}}_{\mathrm{C}}$. □

The $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ semantics essentially extends the $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$ with the sufficient condition (4.28) of subsumption, and shares the other features of $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$: $\mathsf{instanceOf}$ and $\mathsf{subClassOf}$ can be simple roles, subsumption between named concepts is metamodelled fully, the metamodel also keeps track of unnamed subconcepts of named concepts and satisfies both (4.27) and (4.28) accordingly (thus making (4.29) equivalent to $\exists\mathsf{instanceOf}.\mathsf{Kingdom} \sqsubseteq \mathsf{Organism}$).

Also similarly to $\mathcal{HIRS}_{\mathrm{NN}}(\mathcal{SROIQ})$, the $\mathsf{subClassOf}$ role in $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ does not properly metamodel subsumption of unnamed concepts. For them, neither transitivity, the necessary condition (4.27), nor the sufficient condition (4.28) are satisfied in general. Thus again, (4.21) $\not\models$ (4.23), and (4.21, 4.24) $\not\models$ (4.25).

The $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ semantics of $\mathsf{subClassOf}$ can be achieved by a reduction to $\mathcal{SROIQ}$ based on one devised by Glimm et al. (2010) in a less general context.

Definition 4.11 ($\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ First-Order Reduction)**.** *Let $\mathcal{K}$ be a $\mathcal{HIRS}_{\mathrm{SN}}(\mathcal{SROIQ})$ KB in a vocabulary $N = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$. The $\mathcal{SROIQ}$* reduction $\mathcal{K}^{1\mathrm{SN}}$ of $\mathcal{K}$ *is the $\mathcal{SROIQ}$ KB $\mathcal{K}^{1\mathrm{SN}} := \mathcal{K}^{1} \cup \mathrm{SubclassSync}(\mathcal{K})$, where the $\mathcal{SROIQ}$*

KB $\mathcal{K}^1$ in the vocabulary N^1 is the first-order reduction of $\mathcal{K}$ as defined in Def. 4.4, and SubclassSync($\mathcal{K}$) *consists of the following axioms for every $A \in N_\mathrm{C}$:*

$$\begin{aligned} \exists \mathsf{subClassOf}.\top &\sqsubseteq \top_\mathrm{C} \\ \top &\sqsubseteq \forall \mathsf{subClassOf}.\top_\mathrm{C} \end{aligned} \tag{4.37}$$

$$\exists \mathsf{subClassOf}.\{\mathsf{i}_A\} \equiv \forall \mathsf{instanceOf}^-.A. \tag{4.38}$$

Theorem 5. *For any $\mathcal{HIRS}_\mathrm{SN}(\mathcal{SROIQ})$ KB $\mathcal{K}$ and any axiom φ in a common vocabulary N, we have $\mathcal{K} \models \varphi$ iff $\mathcal{K}^{1\mathrm{SN}} \models \mathrm{Int}(\varphi)$.*

Proof. ($\Leftarrow$): Assume $\mathcal{K}^{1\mathrm{SN}} \models \mathrm{Int}(\varphi)$ and take any model $\mathcal{I}$ of $\mathcal{K}$. Let $\mathcal{J} := (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be a first-order interpretation of N^1 defined exactly as in the ($\Leftarrow$) direction of the proof of Theorem 2. From the proof of Thm. 2 follows that $\mathcal{J}$ satisfies Int($\mathcal{K}$), InstSync($\mathcal{K}$) and Typing($\mathcal{K}$). Moreover, from the proof of Thm. 2 also follows that $R^\mathcal{J} = R^{\mathcal{I}\mathcal{E}}$ for all $R \in N_\mathrm{R}$.

Let us show that $\mathcal{J}$ satisfies also SubclassSync($\mathcal{K}$). Clearly, axioms (4.37) are satisfied due to the definition of $\top_\mathrm{C}^\mathcal{J}$ and the fact that $\mathsf{subClassOf}^\mathcal{J} = \mathsf{subClassOf}^{\mathcal{I}\mathcal{E}} \subseteq \Delta_\mathrm{C}^\mathcal{I} \times \Delta_\mathrm{C}^\mathcal{I}$. As for axioms (4.38), for any given $A \in N_\mathrm{C}$, we have

$$\begin{aligned} (\exists \mathsf{subClassOf}.\{\mathsf{i}_A\})^\mathcal{J} &= \{\, c \mid (c, \mathsf{i}_A^\mathcal{J}) \in \mathsf{subClassOf}^\mathcal{J} \,\} \\ &= \{\, c \mid (c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{I}\mathcal{E}} \,\} \\ &\overset{(*)}{=} \{\, c \mid c \in \Delta_\mathrm{C}^\mathcal{I} \wedge c^\mathcal{E} \subseteq A^{\mathcal{I}\mathcal{E}} \,\} \\ &= \{\, c \mid c \in \Delta_\mathrm{C}^\mathcal{I} \wedge \forall x (x \in c^\mathcal{E} \Rightarrow x \in A^{\mathcal{I}\mathcal{E}}) \,\} \\ &= \{\, c \mid \forall x ((x, c) \in \mathsf{instanceOf} \Rightarrow x \in A^{\mathcal{I}\mathcal{E}}) \,\} \\ &= \{\, c \mid \forall x ((x, c) \in \mathsf{instanceOf} \Rightarrow x \in A^\mathcal{J}) \,\} \\ &= (\forall \mathsf{instanceOf}^-.A)^\mathcal{J}, \end{aligned}$$

where $(*)$ holds by Def. 4.10(2). The rest of the proof for this direction is analogous to the ($\Leftarrow$) direction of the proof of Thm. 2.

($\Rightarrow$): Assume $\mathcal{K} \models \varphi$, and take any $\mathcal{J} \models \mathcal{K}^{1\mathrm{SN}}$. Let $\mathcal{I} := (\Delta^\mathcal{I}, \cdot^\mathcal{I}, \cdot^\mathcal{E})$ be an $\mathcal{HIR}(\mathcal{SROIQ})$ interpretation of N defined exactly as in the ($\Rightarrow$) direction of the proof of Thm. 2. From the proof of Thm. 2 follows that $R^{\mathcal{I}\mathcal{E}} = R^\mathcal{J}$ for all $R \in N_\mathrm{R}$. We will show that $\mathcal{I}$ is also a $\mathcal{HIRS}_\mathrm{SN}(\mathcal{SROIQ})$ interpretation by verifying conditions from Def. 4.10.

The first condition (4.10(1)) holds due to the definition of $\Delta_\mathrm{C}^\mathcal{I}$ and (4.37). As for the second condition (4.10(2)), take any $A \in N_\mathrm{C}$ and $c \in \Delta^\mathcal{I}_\mathrm{C}$. Then:

$$\begin{aligned}
&c \in \Delta^\mathcal{I}_\mathrm{C} \wedge (c, A^\mathcal{I}) \in \mathsf{subClassOf}^{\mathcal{IE}} \\
&\quad \text{iff (from the definition of } \Delta^\mathcal{I}_\mathrm{C}) \; c \in \top^\mathcal{J}_\mathrm{C} \wedge (c, \mathsf{i}^\mathcal{J}_A) \in \mathsf{subClassOf}^\mathcal{J} \\
&\quad \text{iff } c \in \top^\mathcal{J}_\mathrm{C} \cap (\exists \mathsf{subClassOf}.\{\mathsf{i}_A\})^\mathcal{J} \\
&\quad \text{iff (from (4.38)) } c \in \top^\mathcal{J}_\mathrm{C} \cap (\forall \mathsf{instanceOf}^-.A)^\mathcal{J} \\
&\quad \text{iff } c \in \top^\mathcal{J}_\mathrm{C} \wedge \forall x((x, c) \in \mathsf{instanceOf}^\mathcal{J} \Rightarrow x \in A^\mathcal{J}) \\
&\quad \text{iff (from the definition of } \Delta^\mathcal{I}_\mathrm{C} \text{ and } c^\mathcal{E}) \; c \in \Delta^\mathcal{I}_\mathrm{C} \wedge \forall x(x \in c^\mathcal{E} \Rightarrow x \in A^{\mathcal{IE}}) \\
&\quad \text{iff } c \in \Delta^\mathcal{I}_\mathrm{C} \wedge c^\mathcal{E} \subseteq A^{\mathcal{IE}}.
\end{aligned}$$

The rest of the proof is analogous to the ($\Rightarrow$) direction of the proof of the Thm. 2. □

Interestingly, combining $\mathcal{HIRS}_\mathrm{SN}(\mathcal{SROIQ})$ with $\mathcal{HIRS}_\mathrm{NA}(\mathcal{SROIQ})$ yields a DL where subClassOf is transitive on all concepts and reflexive on named concepts, at the expense of instanceOf and subClassOf becoming non-simple. This fixes the non-entailments discussed above: (4.21) $\models$ (4.23), and (4.21, 4.24) $\models$ (4.25).

But the sufficient condition of subsumption (4.28) still does not hold for unnamed superconcepts. Take an example:

$$\begin{aligned}
\top &\sqsubseteq {=}1\mathsf{U}.S \\
C &\sqsubseteq \exists \mathsf{instanceOf}.S
\end{aligned} \tag{4.39}$$

in which S is made a singleton concept by the first axiom, and C is made the subclass of the only unnamed instance of S at the extensional level. The problem is that (4.39) does not entail $C\colon \exists \mathsf{subClassOf}.S$. That is, the subclass-of relation is not entailed at the meta level. Semantically, this can be fixed in the analogously defined logic $\mathcal{HIRS}_\mathrm{SA}(\mathcal{SROIQ})$ in which subClassOf has the full set-theoretical semantics also for unnamed concepts, though we do not currently know whether such logic is decidable.

4.5 Type Hierarchy

In $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$, concepts are fully promiscuous, possibly having individuals, roles, and concepts as instances, though about each instance we are able to say if it is an individual, concept, or role. There are certain concepts such as Deprecated for which the promiscuity makes sense. Concepts, roles, and even individuals may become deprecated if they are replaced by new names or more refined versions. In the taxonomy domain, taxa, ranks, and even specimens may become deprecated (e.g., when they are invalidated by further studies). For example, the following axiom states that Cervus camelopardalis (an old binomial name for the species Giraffa camelopardalis) and

Nixus (an old name for the rank Order) are deprecated. Note that Cervus camelopardalis has only individuals as instances, while Nixus has only concepts as instances.

$$\begin{aligned} \textsf{Cervus camelopardalis} &: \textsf{Deprecated} \\ \textsf{Nixus} &: \textsf{Deprecated} \end{aligned} \tag{4.40}$$

An example of a role with a truly promiscuous domain is definedBy (4.4), as it is applicable to both taxa and ranks.

In some cases, promiscuity of all concepts is not desired – e.g., we would like to ensure that concepts Person and Museum classify only individuals, and Species classifies only concepts with individual instances. For cases like this one, we introduce a typing framework axiomatization:

Definition 4.12 (Typing framework)**.** *Given $n \in \mathbb{N}$, a $\mathcal{HIR}(\mathcal{L})$ ($\mathcal{HIRS}_*(\mathcal{L})$)* KB with *n* types *adds new concept names $\top^{X(i)}$ for each i, $0 < i \leq n$, and each $X \in \{\mathrm{I}, \mathrm{R}, \mathrm{IR}\}$, and contains the following axioms for all $X, Y \in \{\mathrm{I}, \mathrm{R}, \mathrm{IR}\}$ and $Z \in \{\mathrm{I}, \mathrm{R}\}$ such that $X \neq Z$:*

1. $\top^{X(t)} \sqsubseteq \forall \mathrm{instanceOf}^-.\top^{X(t-1)}$ *for each t such that $0 < t \leq n$,*
2. $a\colon \top^{\mathrm{I}(1)},\ R\colon \top^{\mathrm{R}(1)}$ *for each $a \in N_\mathrm{I}$, $R \in N_\mathrm{R}$,*
3. $\top^{X(t)} \sqsubseteq \neg\top^{Y(u)}$ *for each $t \neq u$ such that $0 < t, u \leq n$,*
4. $\top^{X(t)} \sqsubseteq \neg\top^{Z(t)}$ *for each t such that $0 < t \leq n$,*
5. $\top^{\mathrm{IR}(1)} \equiv \top^{\mathrm{I}(1)} \sqcup \top^{\mathrm{R}(1)}$.

Axioms 4.12(1) ensure that concepts on level t classify only members of level $t-1$, axioms 4.12(2) assert that all named individuals and roles belong to the lowest level, axioms 4.12(3) and 4.12(4) ensure that different types are disjoint and axiom 4.12(5) establishes the relationship between individual, role and mixed type.

More specifically, the $\top^{\mathrm{I}(1)}$ concept classifies precisely all individuals (similarly, $\top^{\mathrm{R}(1)}$ classifies precisely all roles and $\top^{\mathrm{IR}(1)}$ classifies precisely all individuals and roles), $\top^{\mathrm{I}(2)}$ classifies precisely all concepts with only individual instances (analogously, $\top^{\mathrm{R}(2)}$ classifies all concepts of roles, and $\top^{\mathrm{IR}(2)}$ classifies all first-order concepts), etc. We can thus assert some typing in our example:

$$\begin{aligned} \textsf{Organism} \sqcup \textsf{Person} \sqcup \textsf{Museum} &\sqsubseteq \top^{\mathrm{I}(1)} \\ \textsf{Taxon} &\sqsubseteq \top^{\mathrm{I}(2)} \\ \textsf{Rank} &\sqsubseteq \top^{\mathrm{I}(3)}. \end{aligned}$$

Typing is propagated to subconcepts and instances: Giraffa camelopardalis $\sqsubseteq \top^{\mathrm{I}(1)}$ and Species $\sqsubseteq \top^{\mathrm{I}(2)}$ is now entailed, and so for other taxa and ranks. Domains and ranges of roles may be typed similarly, e.g., $\exists\mathsf{successorOf}.\top \sqsubseteq \top^{\mathrm{I}(2)}$ and $\top \sqsubseteq \forall\mathsf{successorOf}.\top^{\mathrm{I}(2)}$. This, though, is also already entailed, since Species was already asserted as the domain and range (4.5) of the role successorOf, and it is already typed.

Chapter 5

Discussion

In this chapter, we discuss the properties of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ (Sections 5.1 and 5.2), their relationship with set theory and the logic $\mathcal{L}$ (Sections 5.3 and 5.4) and we compare our contribution to other higher-order description logics (Section 5.5).

5.1 Intensional Regularity

In Chapter 3 we already informally introduced the notion of intensional regularity. Now we define it formally.

Definition 5.1 (Intensional regularity)**.** *A higher-order description logic $\mathcal{L}$ has the property of* intensional regularity *if the following implication holds for each knowledge base $\mathcal{K}$ and concepts X, Y in logic $\mathcal{L}$:*

$$\mathcal{K} \models X = Y \implies \mathcal{K} \models X \equiv Y.$$

Intensional regularity is a basic property of HiLog-based logics, and since the semantics of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ is based on HiLog, both are intensionally regular.

Theorem 6. *$\mathcal{HIR}(\mathcal{L})$ and the $\mathcal{HIRS}_*(\mathcal{L})$ variants are intensionally regular.*

Proof. Let $\mathcal{K} \models A = B$ for two concept names A and B, then in every model $\mathcal{I}$ we have $A^\mathcal{I} = B^\mathcal{I}$. Hence also $A^{\mathcal{I}\mathcal{E}} = B^{\mathcal{I}\mathcal{E}}$ and $\mathcal{K} \models A \equiv B$. □

Intensional regularity for concepts is a quite natural requirement for metamodelling (see also Motik (2007)). For example, if we assert that an international and a Slovak name denote the same species (Giraffa camelopardalis = Žirafa štíhla), we also expect their extensions to be equal.

5.2 The Lack of Extensionality

The property of extensionality was also already informally introduced in Chapter 3.

Definition 5.2 (Extensionality)**.** *A higher-order description logic $\mathcal{L}$ has the property of* extensionality *if the following implication holds for each knowledge base $\mathcal{K}$ and concepts X, Y in logic $\mathcal{L}$:*

$$\mathcal{K} \models X \equiv Y \implies \mathcal{K} \models X = Y\,.$$

Another basic property of HiLog-based logics is the *lack* of extensionality, i.e., $\mathcal{K} \models X \equiv Y \;\not\Longrightarrow\; \mathcal{K} \models X = Y$.

Theorem 7. *$\mathcal{HIR}(\mathcal{L})$ and the $\mathcal{HIRS}_*(\mathcal{L})$ variants lack extensionality.*

Proof. Since the extension function is not injectve, a model of a KB such that $\mathcal{K} \models A \equiv B$ can assign A and B distinct intensions $A^{\mathcal{I}} = a \neq b = B^{\mathcal{I}}$ with the same extension, e.g., $a^{\mathcal{E}} = b^{\mathcal{E}} = \{x\}$. □

The lack of extensionality enables, e.g., deprecating an old binomial name of a species without deprecating its newer name, although they classify the same organisms:

$$\begin{gathered}\mathsf{Cervus\ camelopardalis} : \mathsf{Deprecated}\\ \mathsf{Giraffa\ camelopardalis} \equiv \mathsf{Cervus\ camelopardalis}\\ \not\Longrightarrow\ \mathsf{Giraffa\ camelopardalis} : \mathsf{Deprecated}\,,\end{gathered} \tag{5.1}$$

or modelling of single-species genera such as $\mathsf{Sommeromys} \equiv \mathsf{Sommeromys\ macrorhinos}$, where $\mathsf{Sommeromys\ macrorhinos} : \mathsf{Species}$ and $\mathsf{Sommeromys} : \mathsf{Genus}$ without contradicting the disjointness of ranks $\mathsf{Species} \sqcap \mathsf{Genus} \sqsubseteq \bot$.

Note that extensionality and intensional regularity combined mean that concept and role names are unambiguously represented by their extensions. The two properties together erase the distinction between intensions and extensions – concept/role intensions a and b are equal if and only if their extensions $a^{\mathcal{E}}$ and $b^{\mathcal{E}}$ are equal.

We consider the lack of extensionality desirable in metamodelling and we argue for the idea of intensions and extensions as defined by Carnap (1947), already mentioned in Section 4.4: An intension is what a name *means* (the "property" or "character" of a name) and extension is what a name *denotes* (the "class" corresponding to the name). When the state of the world changes, the intension of a name remains the same, but the extension of a name can change. A notorious example are the names Human and Featherless Biped – even though in the current state of world their extensions are the same, their "characters" differ and it is easy to imagine a world where their extensions are different. (And it does not even have to involve Diogenes's plucked chicken (Laërtius and Hicks, 1925).)

5.3 Relationship with Set Theory

Unlike $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$, axiomatic set theories (e.g., ZFC (Shoenfield, 1977) and NBG (Mendelson, 1997)) have both the property of intensional regularity and the property of extensionality. Moreover, $\mathcal{HIR}(\mathcal{SROIQ})$'s ($\mathcal{HIRS}_*(\mathcal{SROIQ})$'s) expressivity makes it vulnerable to Russel's paradox of naïve set theory, which is avoided by axiomatic set theories. A concept of such concepts which are not instances of themselves is defined as $\mathsf{Barber} \equiv \neg\exists\mathsf{instanceOf.Self}$. Take any $\mathcal{HIR}(\mathcal{SROIQ})$ ($\mathcal{HIRS}_*(\mathcal{SROIQ})$) model $\mathcal{I}$ of $\mathcal{K}$, and let

$$b := \mathsf{Barber}^{\mathcal{I}} \in \Delta^{\mathcal{I}}_{\mathrm{C}}$$
$$B := b^{\mathcal{E}}$$
$$S := \exists\mathsf{instanceOf.Self} = \{\, x \mid (x, x) \in \mathsf{instanceOf}^{\mathcal{E}} \,\} = \{\, x \mid x \in \Delta^{\mathcal{I}}_{\mathrm{C}} \wedge x \in x^{\mathcal{E}} \,\}.$$

We have $B = \Delta^{\mathcal{I}} \setminus S = \Delta^{\mathcal{I}}_{\mathrm{I}} \uplus \Delta^{\mathcal{I}}_{\mathrm{R}} \uplus (\Delta^{\mathcal{I}}_{\mathrm{C}} \setminus S) = \Delta^{\mathcal{I}}_{\mathrm{I}} \uplus \Delta^{\mathcal{I}}_{\mathrm{R}} \uplus \{\, x \mid x \in \Delta^{\mathcal{I}}_{\mathrm{C}} \wedge x \notin x^{\mathcal{E}} \,\}$. Hence the contradiction: $b \in b^{\mathcal{E}}$ iff $b \notin b^{\mathcal{E}}$. However, this example is actually not specific to $\mathcal{HIR}(\mathcal{SROIQ})$ or $\mathcal{HIRS}_*(\mathcal{SROIQ})$ logics, as it reduces to a contradictory $\mathcal{SROIQ}$ knowledge bases. The reduced knowledge base obtained from $\mathcal{HIR}(\mathcal{SROIQ})$ contains the following axioms:

$$\mathsf{Barber} \equiv \neg\exists\mathsf{instanceOf.Self} \tag{5.2}$$
$$\mathsf{Barber} \equiv \exists\mathsf{instanceOf}.\{\mathsf{i_{Barber}}\}, \tag{5.3}$$

which suffice to create the contradiction in the reduced knowledge base:

$$\begin{aligned}
&(\mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}}, \mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}}) \in \mathsf{instanceOf}^{\mathcal{I}} \\
&\quad \text{iff (from (5.3)) } \mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}} \in \mathsf{Barber}^{\mathcal{I}} \\
&\quad \text{iff (from (5.2)) } \mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}} \notin (\exists\mathsf{instanceOf.Self})^{\mathcal{I}} \\
&\quad \text{iff } (\mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}}, \mathsf{i}^{\mathcal{I}}_{\mathsf{Barber}}) \notin \mathsf{instanceOf}^{\mathcal{I}}.
\end{aligned}$$

In $\mathcal{HIR}(\mathcal{L})$ ($\mathcal{HIRS}_*(\mathcal{L})$) (where $\mathcal{L}$ admits GCIs, existential and universal restriction, qualified number restriction, role inverses and simple RIAs), it is also possible to construct infinite descending chains of instantation. Such chains are in axiomatic set theories avoided by the axiom of foundation (Barwise and Moss, 1996). They can be constructed in our logics by creating a chain with a new simple role S, concept X and individual z by the usual axioms $X \sqsubseteq \exists S.X$, $X \sqsubseteq \leqslant 1 S^-.X$, $z\colon X \sqcap \forall S^-.\bot$, and then asserting $S^- \sqsubseteq \mathsf{instanceOf}$ (see Figure 5.1). While some approaches to metamodelling (Pan et al., 2005; Motz et al., 2015) avoid or explicitly prohibit such chains, the reducibility of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ to $\mathcal{L}$ (or $\mathcal{LO}$) means that they do not cause any harm regarding decidability. Moreover, both the construction of infinite descending instantiation chains and of Russel's paradox can be prevented by using the typing framework from Section 4.5.

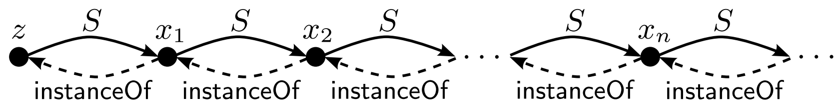

Figure 5.1: Infinite Instantiation Chain

5.4 Relationship with the Logic $\mathcal{L}$

Since both $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ are reducible to the logic $\mathcal{L}$ (or $\mathcal{LO}$ in case that $\mathcal{L}$ does not feature nominals), the expressivity of $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ is actually the same as the expressivity of $\mathcal{L}$ ($\mathcal{LO}$). At first, this might seem a bit counterintuitive, however, the reductions prove that our extensions of $\mathcal{L}$ are only syntactic sugar allowing the user to easily model with higher orders and even with the relationships of instantiation and subsumption. Yet, we consider our results important, because they show *how* expressive $\mathcal{L}$ (or, in the case we explored the most, $\mathcal{SROIQ}$) actually is.

5.5 Comparison with Other Higher-Order DLs

While in some other higher-order DLs (e.g., Motik (2007) and De Giacomo et al. (2011) mentioned in the previous chapter) any name simultaneously denotes an individual, a concept, and a role, $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ keep the basic distinction among these three types of entities. Individuals have no extensions, concepts only have concept extensions (a set of entities they classify), and roles only have role extensions (a set of pairs of entities they interconnect). This threefold distinction is fundamental from the ontological standpoint (see, e.g., Grossmann (1983, Pt. II), Guizzardi (2005, Ch. 4, 6, 7), Svátek et al. (2013)) and dates back to Aristotle (Aristotle, 1962, 2a11,6a37). It also saves users accustomed to first-order DLs from surprises, and provides basic sanity checks.

Intensional regularity and the lack of extensionality in $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ are both inherited from HiLog (Chen et al., 1993), and thus shared by its other descendants (ν-semantics (Motik, 2007), $Hi(\mathcal{SHIQ})$ (De Giacomo et al., 2011) and $\mathcal{TH}(\mathcal{SROIQ})$ (Homola et al., 2014)).

There are use cases where extensionality is needed, as demonstrated by Motz et al. (2015). Such cases can be covered by one of the logics with truly higher-order semantics, which domains contain not only basic objects, but also sets of basic object, sets of sets of basic objects, etc. (OWL FA (Pan et al., 2005), $\mathcal{SHIQM}$ (Motz et al., 2015)), though none features metamodelling of instantiation or subsumption. Since punning in OWL 2 (Cuenca Grau et al., 2008) is semantically equivalent to Motik's π-semantics, it is neither intensionally regular nor extensional (Motik, 2007). E.g., a knowledge

Table 5.1: Comparison of Properties of Higher-Order Description Logics

	Unlimited higher orders	Typing	Concept/role promiscuity	Intensional regularity	Extensionality	Instantiation metamodelling	Subsumption metamodelling	Expressivity
OWL FA	Y	Y	N	Y	Y	N	N	$\mathcal{SHOIQ}$
ν-semantics	Y	N	Y	Y	N	N	N	$\mathcal{ALCHOIQ}$
Punning	Y	N	N	N	N	N	N	$\mathcal{SROIQ}$
$Hi(\mathcal{SHIQ})$	Y	N	Y	Y	N	N	N	$\mathcal{SHIQ}$
$\mathcal{O}^{\text{meta}}$	N	Y	N	Y	N	Y	Y*	$\mathcal{SROIQ}$
$\mathcal{SHIQM}$	Y	Y*	Y	Y	Y	N	N	$\mathcal{SHIQ}$
$\mathcal{TH}(\mathcal{SROIQ})$	Y	Y	N	Y	N	N	N	$\mathcal{SROIQ}$
$\mathcal{HIR}(\mathcal{SROIQ})$	Y	Y	Y	Y	N	Y	N	$\mathcal{SROIQ}$
$\mathcal{HIRS}_*(\mathcal{SROIQ})$	Y	Y	Y	Y	N	Y	Y*	$\mathcal{SROIQ}$

base $\mathcal{K} = \{\mathsf{x} \colon \mathsf{A}, \mathsf{A} = \mathsf{B}, \mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot\}$ entails that names A and B represent the same individual, but distinct classes. However, $\mathcal{K}$ is consistent, because the interpretations of A and B are determined by their contexts (whether the name denotes an individual or a class) and these interpretations are independent. Punning thus provides only very basic support for metamodelling via, essentially, overloading of names. It can, however, still be useful in some applications (Noy, 2005).

In Table 5.1, we compare key properties of our logics $\mathcal{HIR}(\mathcal{SROIQ})$ and $\mathcal{HIRS}_*(\mathcal{SROIQ})$ with other higher-order description logics. The table does not contain information about complexity of different logics on purpose – their complexities are the same as the complexities of the underlying description logics, with the exception of OWL FA and $\mathcal{SHIQM}$, which complexities are not known.

Note the asterisks by some of the yes/no answers: The encoding scheme $\mathcal{O}^{\text{meta}}$ (Glimm et al., 2010) featured role subClassOf which was axiomatized similarly as subClassOf in our $\mathcal{HIRS}_{\text{SN}}(\mathcal{SROIQ})$ (since Glimm et al. (2010) used first-order semantics, they did not encounter any limitations on the subsumption of unnamed concepts).

In case of $\mathcal{SHIQM}$ (Motz et al., 2015) and typing, the logic's semantics incorporates well-founded sets and thus it does not allow an instanceOf-cycle or an infinite descending chain. However, the type hierarchy is "automatic" and cannot be explicitly enforced on a concept or a role by the user. Thus, it cannot be used to organize concepts and roles into layers, like our typing framework from Section 4.5.

Finally, $\mathcal{HIRS}_*(\mathcal{SROIQ})$ features different axiomatizations of subClassOf with different properties and limitations. We have argued in Section 4.4 that they are all useful and sufficient for most of the metamodelling use cases, but none of them has set-theoretic semantic on all (named and unnamed) concepts.

Conclusions

In this thesis, we have introduced and studied extensions of description logic (dubbed $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ for DL $\mathcal{L}$ admitting GCIs, existential restriction and complement, and in case of $\mathcal{HIRS}_{\text{NA}}(\mathcal{L})$ also complex RIAs). They feature domain metamodelling in the form of unlimited higher orders and full metamodelling of instantiation and partially also subsumption.

We proved decidability of our extensions by means of reduction to the logic $\mathcal{L}$ (or, in some cases, $\mathcal{LO}$). This reduction shows that the expressive power needed to model with higher orders and metamodel instantiation and partially also subsumption relationship was already present in the logic $\mathcal{L}$ ($\mathcal{LO}$). Since $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ can be reduced to the base logic $\mathcal{L}$ (or $\mathcal{LO}$), it is possible to decide them using off-the-shelf reasoners. Moreover, since the reduction is polynomial, deciding $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ can be done with the same computational complexity as deciding the base logic $\mathcal{L}$ (or $\mathcal{LO}$).

Further, we showed that our approach has properties (intensional regularity and the lack of extensionality) and features (instantiation and subsumption metamodelling, unlimited higher orders, type hierarchy, concept and role promiscuity) desirable for metamodelling and that such combination of attributes is not present in any previous work on metamodelling. $\mathcal{HIR}(\mathcal{L})$ and $\mathcal{HIRS}_*(\mathcal{L})$ thus allow the user to easily and correctly model domains with inherent higher-order structure, such as the biological taxonomy. Moreover, they also allow the user to freely metamodel with instantiation and partially also with subsumption.

While we showed three possible axiomatizations of subsumption, the problem of deciding set-theoretic subsumption metamodelling also for unnamed concepts remains unsolved and might be an interesting direction for future work. Another possible question for future research is if such domain and full metamodelling features could be added to lesser expressive description logics.

List of Figures

List of Tables

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