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Circular Chromatic Index of Snarks

Diploma Thesis

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Abstract

A circular r-edge-coloring of a graph G is a mapping $c : E(G) \to [0, r)$ such that for any two adjacent edges e and f of G we have $1 \leq |c(e) - c(f)| \leq r - 1$. The circular chromatic index $\chi'_c(G)$ is the infimum of all real numbers r such that G has a circular r-edge-coloring.

We establish a general lower bound for the circular chromatic index of a snark G depending only on the order of G. This bound is asymptotically tight. We also determine the exact value of the circular chromatic index of the generalized Blanuša snarks. In this case, the index takes infinitely many values and can be arbitrarily close to 3. The generalized Blanuša snarks are the first explicit class of snarks with this property.

Keywords: snark, Blanuša snark, circular chromatic index

Contents

Introduction

Graph coloring problems are known for a long time. The most famous among them is the Four-Color Problem. Many approaches have been used while tackling this problem. P. G. Tait initiated the study of edge colorings by proving that the Four-Color Theorem is equivalent to the statement that every bridgeless cubic planar graph is 3-edge-colorable. Bridgeless cubic graphs which are not 3-edge-colorable were named *snarks* by Martin Gardner, because they are rather rare and people were hunting for them: a planar snark would be a counterexample to the Four-Color Theorem.

Many other connections between snarks and other areas of the graph theory are known. There are several conjectures to which the smallest counterexample is known to be a snark. The most famous of them is the Cycle Double Cover Conjecture, which is still open and motivating research today.

In a classical 3-edge-coloring of a cubic graph we use three colors. The colors are naturally represented as integers 0, 1, 2. There is no reason to impose such a restriction. There are two ways of a natural generalization: the colors could be elements of an additive group or real numbers. The group approach leads to flows. A variant of the second approach, to admit the colors to be fractions, was introduced by Vince in [6] under the name *star coloring*. Currently it is widely known as *circular coloring*.

Vizing's theorem says that graphs can be divided into two classes – those with chromatic index equal to the maximal degree Δ and those which require one more color. Circular chromatic index is a refinement of the usual chromatic index. In some sense it says which snarks are closer to being 3-edge-colorable than the others.

In general, very little is known about the circular chromatic index, and this remains true if we restrict ourselves to snarks. The aim of this work is to find a general lower bound on the circular chromatic index and to use it to determine the circular chromatic index of the Blanuša snarks.

Chapter 1

Circular Chromatic Index

1.1 Colors as Real Numbers

Edge colorings are a special case of vertex colorings, and this remains true for circular colorings. Many properties of vertex colorings can be carried over to edge colorings without any problem, and it is simpler to prove them for vertex colorings. This is the reason why we begin with vertex colorings instead of beginning with edge colorings directly.

Although we assume the reader to be familiar with the usual vertex colorings of graphs, we give the precise definition.

Definition 1. An *r-coloring* of a graph G is a mapping $c : V(G) \to \{0, 1, \dots, r-1\}$ such that for any two adjacent vertices u and v we have

$$1 \leq |c(u) - c(v)| \leq r - 1.$$

The least integer r such that the graph G has an r-coloring is called *chromatic number* and is denoted by $\chi(G)$. We say that a graph G is *r-colorable* if there is an r-coloring of G.

Why do we use integers as colors? There is no reason to impose this restriction. We allow ourselves to use real numbers as colors. Let us color the vertices of a graph G by reals from the interval $[0, r)$. Now we have a few issues to deal with.

We see that in the usual coloring we have "different" colors of adjacent vertices. This means that the absolute value of their difference is at least 1. This is exactly what we want from our new coloring by real numbers. The largest difference of colors in the usual coloring is $r-1$. We impose this restriction also on the new coloring. Now we are in a position to define the circular coloring precisely. (The word "circular" will be explained later.)

Definition 2. A *circular r-coloring* of a graph G is a mapping $c : V(G) \to [0, r)$ such that for any two adjacent vertices u and v we have

$$1 \leq |c(u) - c(v)| \leq r - 1.$$

The *circular chromatic number*, denoted by $\chi_c(G)$, is defined by

$$\chi_c(G) = \inf \{r : G \text{ has a circular } r\text{-coloring}\}.$$

We say that a graph G is *circularly r-colorable* if there is a circular r-coloring of G.

The circular chromatic number is well-defined – if a graph G has a circular r-coloring, then it has circular r'-coloring for any $r' > r$. The circular r-coloring is just the usual r-coloring, if we use only integers from $[0, r)$ as colors. This suggests that the circular chromatic number is not greater than the usual chromatic number. This, and even more, is captured in the following theorem.

Theorem 3. *For any graph G we have $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. In other words, $\chi(G) = \lceil \chi_c(G) \rceil$.*

Proof. Assume there exist a circular r-coloring c of G such that $r \leq \chi(G) - 1$. Let $c'(v) = \lfloor c(v) \rfloor$ for any vertex $v \in V(G)$. Now $c'(v)$ is an integer and $1 \leq |c'(u) - c'(v)| \leq r - 1$ holds for any $u, v \in V(G)$. Thus we obtained an $\lfloor r \rfloor$-coloring c' of the graph G. We have $\lfloor r \rfloor \leq r \leq \chi(G) - 1$, hence we have a $(\chi(G) - 1)$-coloring of G, which is impossible. The second inequality is obvious. □

The infimum in the definition of the circular chromatic number is always attained. The circular chromatic number is rational for any finite graph G. (This is not true for infinite graphs, but such matters are beyond the scope of this work. All graphs considered here are finite and simple.) Both of these properties of the circular chromatic number were first proved by Vince in [6] by methods of continuous mathematics; a purely combinatorial proof was given by Bondy and Hell in [7].

1.2 Equivalent Definitions

We have promised to explain the word "circular" in the definition of a circular coloring. Therefore we give an equivalent definition of a circular coloring that has something to do with circles.

Definition 4. Let C be a circle with perimeter $r > 0$. A *circular r-coloring* of a graph G is a mapping f which assigns to each vertex v an open unit-length arc $f(v)$ of C in such a way that for any adjacent vertices u and v we have $f(u) \cap f(v) = \emptyset$.

Let c be a circular r-coloring of G according to Definition 2. We construct a mapping f, which will be a circular r-coloring of G according to Definition 4. Let v be a vertex of G. Let $f(v) = (c(v), c(v) + 1)$ (the right endpoint is taken modulo r). It is easy to verify that arcs corresponding to adjacent vertices have empty intersection.

To prove the second implication of our equivalence, let $c(v)$ be the left endpoint of the arc $f(v)$. Again, it is easy to verify all conditions imposed on the mapping c constructed in this way.

Together we have proved that a graph G has a circular r-coloring according to Definition 2 if and only if G has a circular r-coloring according to Definition 4. The choice of the definition depends on the problem we are tackling. Moreover, we give one more equivalent definition which will be used most often in this work. It is the definition used by Vince to introduce star colorings.

Definition 5 (Vince). Let p and q be positive integers. A *(p, q)-coloring* of a graph G is a mapping $c : V(G) \to \{0, 1, \ldots p - 1\}$ such that for any two adjacent vertices u and v we have

$$q \leq |c(u) - c(v)| \leq p - q.$$

A graph has a circular (p/q)-coloring if and only if it has a (p, q)-coloring. The proof of equivalence can be found in [8].

Remark. This coloring is often referred to as a p/q-coloring instead of a (p, q)-coloring. In fact, all the three given definitions are equivalent, and the usual coloring is a special case of these colorings. Therefore it is no need to distinguish between the usual coloring and the circular coloring. We will use the term "r-coloring" for circular r-coloring. Similarly we say "r-edge-coloring" instead of "circular r-edge-coloring".

1.3 Basic Properties of Circular Edge Colorings

Let G be a graph. Let $L(G)$ be a graph with vertex set $E(G)$ and two vertices u and v of $L(G)$ adjacent if and only if u and v are adjacent as edges is G. The graph $L(G)$ is called the *line graph* of the graph G. It is easy to see that any edge coloring of G has the corresponding vertex coloring of $L(G)$, and vice versa. Therefore we are allowed to define a circular edge coloring as follows.

Definition 6. A *circular r-edge-coloring* of a graph G is a circular r-coloring of $L(G)$. The *circular chromatic index* of a graph G is the circular chromatic number of its line graph; $\chi'_c(G) = \chi_c(L(G))$.

Definition 7. A *(p, q)-edge-coloring* of a graph G is a (p, q)-coloring of $L(G)$.

Our aim is to examine edge colorings. First we state several basic properties of edge colorings. There is no problem to generalize them for vertex colorings. We know that the circular chromatic index of a graph G is a rational number p/q. Moreover, we may assume that p and q are coprime.

Theorem 8. *Let G be a graph with $\chi'_c(G) = p/q$, where p and q are coprime integers.*

a) There exist a (p, q)-edge-coloring of G.

b) In any (p, q)-edge-coloring of G any of the colors $0, 1, \ldots, p-1$ is used at least once.

c) We have $p \leq |E(G)|$.

d) The denominator q is at most the cardinality of maximum matching of G, in particular, $q \leq |V(G)|/2$.

Proof. The parts a) and b) are easy consequences of Lemma 1.3 of [8]. The part c) is obvious from b). To prove d), note that the edges colored by colors $0, 1, \ldots, q-1$ form a matching in G. $\square$

Chapter 2

Snarks

As we have said before, bridgeless cubic graphs that are not 3-edge-colorable are of special interest. They are called *snarks*. Until Isaacs constructed two infinite classes of snarks in [2], there were only four known snarks. Currently we have a lot of families of snarks and some constructions for creating new snarks from existing ones. There have been many attempts to classify snarks in some reasonable way, for example [15]. Some snarks are considered trivial; we will discuss it in the next part.

Questions of Triviality

If a graph has more than one component, we can color the components separately. Therefore we may assume that all considered graphs are connected. It is a consequence of the Parity Lemma (stated and proved later in this chapter) that a cubic graph with a bridge is not 3-edge-colorable. Therefore cubic graphs with bridges are "trivially" snarks and we exclude them from the definition of a snark.

Next we focus on girth. A cubic graph always contains a cycle, so its girth is at least 3.

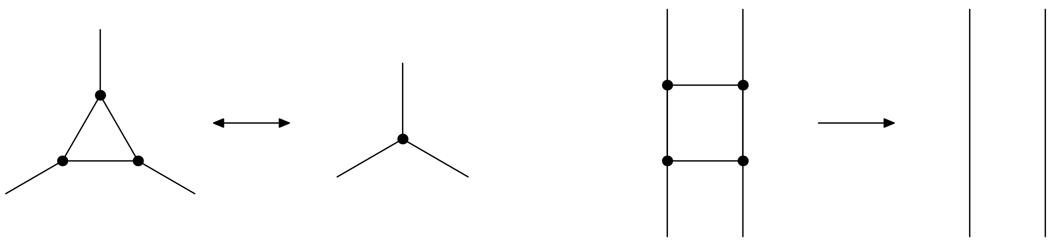

Figure 2.1: Replacing triangles and quadrilaterals in a snark.

Consider a snark G with a triangle. This triangle can be replaced by a vertex to get a smaller snark H (with two vertices less than the original snark). It works also the other way: replacing any vertex in a snark H by a triangle we get another snark G, because 3-edge-colorability of G implies 3-edge-colorability of H and vice versa. Therefore snarks with girth 3 are "trivial" and are excluded from the definition of "genuine" snarks.

Consider a snark G with girth 4. It contains a quadrilateral. This quadrilateral can be replaced by a pair of two parallel edges, in this way we obtain a graph H. Again, if H is 3-edge-colorable, then also G is 3-edge-colorable; a contradiction. Hence H is a snark. This construction (shown in Figure 2.1) does not work in the other way; by replacing a pair of parallel edges by a quadrilateral we can obtain a 3-colorable graph from a snark.

Therefore a snark is considered "nontrivial", if it has the girth at least 5. What can we say about the edge connectivity number of a snark? It is a cubic graph, so there is a cut with 3 edges, that decomposes the graph into two components – we can take three edges adjacent with the same vertex to obtain such a cut. This situation is not interesting, we look for cuts that are not so trivial – cycle-separating cuts. By using the Parity Lemma (see the next section) it is easy to prove that if a snark G has a cycle-separating cut with three edges, then we can take one of the pieces separated by the cut and construct a new snark which is smaller than the original one. Therefore we assume that a "nontrivial" snark is cyclically 4-edge-connected.

There are similar results for higher edge connectivity numbers, and we could demand cyclic 6-edge connectivity from a nontrivial snark, but we stick to the definition most often used in the literature.

Definition 9. A *snark* is a cyclically 4-edge-connected bridgeless cubic graph G with girth at least 5 and $\chi'(G) = 4$.

2.1 Construction of Snarks

The Parity Lemma

Many constructions of snarks are based on the Parity Lemma. The underlying concept of the lemma is that of a *flow*. Therefore we first transform a 3-edge-coloring into a flow and then state and prove the Parity Lemma.

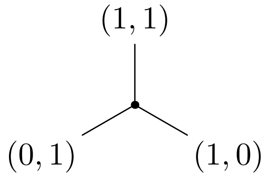

Figure 2.2: A coloring of edges incident with the same vertex.

Consider a 3-edge-coloring of a cubic graph. If we know the colors of two edges incident with the same vertex, we know the color of the third edge incident with that vertex. Hence a 3-edge-coloring is a kind of flow. In fact, we can use nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as colors to obtain a nowhere-zero flow that corresponds to a 3-edge-coloring of our graph. It is easy to verify that for any $a, b, c \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0,0)\}$ we have $a + b + c = 0$ if and only if a, b, c are pairwise distinct. The colors on the edges incident with the same vertex are shown in Figure 2.2.

Lemma 10 (The Parity Lemma). *Let G be a cubic graph with a cutset A consisting of n edges. Consider a 3-edge-coloring of G with colors 1, 2, and 3. Let n_i be the number of edges in A colored by the color i for $i = 1, 2, 3$. Then*

$$n_1 \equiv n_2 \equiv n_3 \equiv n \pmod 2.$$

Proof. As we have shown, we can identify the colors with the nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, obtaining a nowhere-zero flow. Therefore the sum of the colors on the edges of the cutset A is zero:

$$n_1 \times (1,0) + n_2 \times (0,1) + n_3 \times (1,1) = (0,0).$$

Hence

$$\begin{aligned} n_1 + n_3 &\equiv 0 \pmod 2, \\ n_2 + n_3 &\equiv 0 \pmod 2, \end{aligned}$$

and this implies $n_1 \equiv n_2 \equiv n_3 \pmod 2$, hence $n_i \equiv n \pmod 2$. □

The lemma can be proved also by Kempe chains or other flow-free arguments. However, we have seen that a 3-edge-coloring of a cubic graph is in fact a flow and it is pleasant to work with it as with a flow. Nothing like this can be said about circular edge colorings, as we will see in Chapter 3.

Dot Product

The first known construction of snarks is the dot product operation. We take two snarks G_1 and G_2 and form a new snark $G_1 \cdot G_2$ as follows.

1) Remove any two adjacent vertices from G_1.

2) Remove any two nonadjacent edges from G_2.

3) Join vertices a, b, c, d to $1, 2, 3, 4$ in this order (see Figure 2.3).

A result of the dot product depends not only on G_1 and G_2, but also on the choices made in 1) and 2). For example, the two types of the Blanuša snark are formed by different choices from two copies of the Petersen graph.

We claim that we always obtain a snark by the dot product. This deserves a proof. Assume the resulting graph is 3-edge-colorable. We have a cutset of size 4. According to the Parity Lemma edges from this cutset can be colored 1111, 1122, 1212 or 1221. Any of these possibilities allows a 3-edge-coloring of G_1 or G_2, and this is a contradiction.

Other Constructions

There are many other constructions of snarks, some examples are in [3]. Some of them use the Parity Lemma, mainly the general ones. There are certain specific constructions which exclude the possibility of a 3-edge-coloring without the Parity Lemma, for example the construction of flower snarks. Those constructions are based on a case analysis of a small piece of a graph. On the other hand, we have not seen any constructions of graphs with circular chromatic index in a given nontrivial interval.

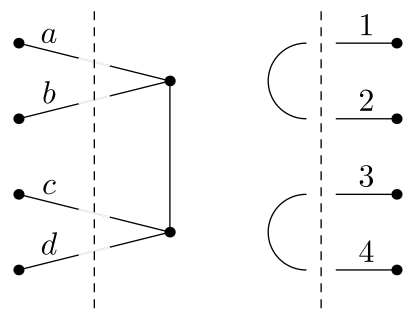

Figure 2.3: Dot product of two cubic graphs.

Chapter 3

Determining the Circular Chromatic Index of a Snark

It is a consequence of the Theorem 3 that the circular chromatic index of a snark lies in the interval $(3, 4]$. The upper bound cannot be attained. It has been proved in [10] that a cubic graph without a bridge has the circular chromatic index at most 11/3. The upper bound is tight, because the circular chromatic index of the Petersen graph is 11/3, as we will show later. We do not know of any other graph G with $\chi'_c(G) > 7/2$. In Chapter 4 we show that the circular chromatic index of a snark can be arbitrarily close to 3.

The existence of snarks of arbitrarily high girth was proved by Kochol in [13]. On the other hand, high girth means small circular chromatic index, as showed in [14]: for any $\varepsilon > 0$ there exist an integer g such that the circular chromatic index of every bridgeless cubic graph with girth at least g is at most $3 + \varepsilon$. In particular every bridgeless cubic graph with girth at least 14 has circular chromatic index at most 7/2.

The question is how to determine the circular chromatic index of a particular graph. It is not at all easy. For small graphs we can do a case analysis by hand or use backtracking by a computer. As the number of vertices grows, also the upper bound for the nominator and denominator grows. There is no known effective algorithm to go through so many possible colorings. It is not surprising: even determining the usual chromatic index of a cubic graph is known to be NP-complete (for a proof see [5]).

There are only a few classes of snarks with known value of circular chromatic index; we will list them later in Section 3.2. Why is it so hard to determine the circular chromatic index? One of the problems seems to be the absence of almost any flow arguments to handle circular colorings. There is nothing like the Parity Lemma for $(p.q)$-edge-colorings. It can even be worse. Consider a $(3k+1,k)$-edge-coloring. If we have a vertex with two edges colored by 0 and k, the third edge incident with this vertex can be colored by $2k$ or $2k+1$. This is the difference between the ordinary 3-edge-coloring and a $(3+\varepsilon)$-edge-coloring: in the 3-edge-coloring we have the color of the third edge determined by the colors of the first two edges and in the $(3+\varepsilon)$-edge-coloring this fails. In a snark with enough vertices there can be any colors on a cut in a $(3+\varepsilon)$-edge-coloring, so there cannot be anything like the Parity Lemma.

In our effort to find a lower bound we use the following simple observations.

- If a cubic graph contains two disjoint perfect matchings, it is 3-edge-colorable.
- Two perfect matchings in a simple graph cannot differ in a single edge.

These observations were used in [11] to determine the circular chromatic index of small flower snarks. We use them to derive a general lower bound for a snark of a given order.

3.1 Lower Bound for a Snark of a Given Order

We start with two technical lemmas.

Lemma 11. *Let $k \geq 3$ be an integer. Consider the fractions p/q with denominator $q \leq 2k+1$ satisfying*

$$\frac{p}{q} > 3 + \frac{1}{k}.$$

The least of all such fractions is $\frac{6k-1}{2k-1} = 3 + \frac{2}{2k-1}$.

Proof. We proceed by contradiction. Assume we have a fraction p/q with denominator $q \leq 2k+1$ satisfying

$$3 + \frac{2}{2k-1} > \frac{p}{q} > 3 + \frac{1}{k}.$$

Then we have a fraction p'/q with denominator $q \leq 2k+1$ satisfying

$$\frac{2}{2k-1} > \frac{p'}{q} > \frac{2}{2k}. \tag{3.1}$$

If $p' \geq 3$, then

$$\frac{2}{2k-1} > \frac{p'}{q} \geq \frac{3}{q} \geq \frac{3}{2k+1}.$$

This implies $4k+2 > 6k-3$ which clearly does not hold for $k \geq 3$.

Otherwise $p' \in \{1,2\}$. Therefore $2q/p'$ is an integer. From (3.1) we have

$$2k-1 < \frac{2q}{p'} < 2k,$$

yielding a contradiction. □

Lemma 12. *Let $k \geq 4$ be an integer. Consider the fractions p/q with denominator $q \leq 2k+1$ satisfying*

$$\frac{p}{q} > 3 + \frac{2}{2k-1}.$$

The least of all such fractions is $\frac{3k-2}{k-1} = 3 + \frac{1}{k-1}$.

Proof. Essentially the same as the proof of Lemma 11. □

Lemma 13. *Let G be a connected cubic graph with $4k+2$ vertices. If G does not have a 3-edge-coloring, then $\chi'_c(G) > 3 + 1/k$.*

Proof. Assume G has a $(3k+1,k)$-edge-coloring c. We derive a contradiction. Graph G has $6k+3$ edges. Assume that one color (say, 0) is used at most once. Then the sets

$$M_1 = c^{-1}(\{1,2,\ldots,k\}), \qquad M_2 = c^{-1}(\{k+1,k+2,\ldots,2k\}),$$
$$M_3 = c^{-1}(\{2k+1,2k+2,\ldots,3k\}).$$

are pairwise disjoint matchings in G. Graph G cannot have two disjoint perfect matchings, because it does not have a 3-edge-coloring. Hence at most one of the matchings M_1, M_2, and M_3 is perfect. So we have

$$\begin{aligned} 6+3k &= |E(G)| = |c^{-1}(\{0\}) \cup M_1 \cup M_2 \cup M_3| \leq 1 + |M_1| + |M_2| + |M_3| \\ &\leq 1 + (2k+1) + 2k + 2k = 6k+2. \end{aligned}$$

This is impossible. Therefore we know that any color is used at least twice which implies that one color (say, $k-1$) is used three times and any other color is used exactly two times. Consider matchings

$$M_1 = c^{-1}(\{0,1,2,\ldots,k-1\}) \qquad \text{and} \qquad M_2 = c^{-1}(\{1,2,\ldots,k\}).$$

The matchings M_1 and M_2 are perfect, because both of them consist of $2k+1$ edges. In M_1 we have two edges v_1v_2 and v_3v_4 colored with color 0. Those edges are not contained in M_2. Any other edge from M_1 is contained in M_2. The matching M_2 is perfect, so there has to be exactly one edge incident with each one of the vertices v_1, v_2, v_3, v_4. Those edges are, say, v_1v_3 and v_2v_4. This means that we have a cycle of length 4 in the graph G. The edges of this cycle are colored 0, k, 0, k in the indicated order. The graph G is cubic, so we have four other edges incident with the vertices v_1, v_2, v_3, v_4 respectively. There are only two possibilities for the colors of those edges: $2k-1$ and $2k$.

If G has girth at least 5, we are ready. Now we prove that G with girth at most 4 is 3-edge-colorable, which is the contradiction sought. We repeat the argument from the previous paragraph. The conclusions are summarized in the following table.

M_1 induced by colors	M_2 induced by colors	cycle	colors of edges
$0, 1, 2, \ldots, k-1$	$1, 2, \ldots, k+0$	$0, k, 0, k$	$2k, 2k+1$
$1, 2, 3, \ldots, k+0$	$2, 3, \ldots, k+1$	$1, k+1, 1, k+1$	$2k+1, 2k+2$
$2, 3, 4, \ldots, k+1$	$3, 4, \ldots, k+2$	$2, k+2, 2, k+2$	$2k+2, 2k+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-2, k-1, \ldots, 2k-3$	$k-1, k, \ldots, 2k-2$	$k-2, \ldots, 2k-2$	$3k-2, 3k-1$

Let C_i be the cycle with edges colored by colors i, $k+i$, i, $k+i$ for $i = 0, 1, \ldots, k-2$. Three edges colored by color $k-1$ are not incident with any vertex from the cycles $C_0, \ldots, C_{k-2}$. No two of these edges have a vertex in common, so they form a graph D on six vertices. Edges colored by the colors $2k-1$ and $3k$ are not incident with any vertex from the cycles $C_0, \ldots, C_{k-2}$, hence their endpoints belong to $V(D)$. We add these edges to D. Now D needs four more edges to be cubic, these edges can be colored only by the colors $3k-1$ and $2k$. Therefore we can look at D as it is C_{k-1}: the edges from C_i can go to C_{i-1} or to C_{i+1}, but to no other C_j (indices are taken modulo $k-1$).

Consider a cycle C_i with vertices v_1, v_2, v_3, v_4 (they form the cycle in this order). Assume that v_1 and v_3 are joined by an edge. Then v_2 and v_4 are not joined by an edge, if they were, C_i would be a proper component of G, but G is connected. So C_i either contains a triangle or not. In the case it contains a triangle both edges leaving C_i are of the same color, that is, C_i is joined to one of its neighbours C_{i-1}, C_{i+1}, but not to both of them. A similar idea can by applied to C_{k-1}: it has four "semiedges", two of them colored by $2k$ and two of them colored by $3k-1$. We can join two semiedges of the same color together or join them both to one of the neighbours C_{k-2} and C_0.

Now we will perform on G two operations showed in Figure 3.1.

1. Replacing a C_i not containing a triangle by a pair of two parallel edges.

2. Replacing a C_i containing a triangle by a single edge.

The first operation is removing a quadrilateral which we have already seen. We have checked that if G is not 3-edge-colorable, then also G' obtained by the first operation from G is not 3-edge-colorable. Look at the second operation. Assume we have applied it on G and obtained a graph G'. One can easily verify that if G' has a 3-coloring, then also G has a 3-coloring. In other words: G' is not 3-edge-colorable, because G is not 3-edge-colorable.

Figure 3.1: Operations 1. (on the left) and 2. (on the right).

Repeating these operations we finally arrive at a graph H. In this graph there are no vertices from $C_0, C_1, \ldots, C_{k-2}$; all were removed either by the first or by the second operation. Thus H has at most 6 vertices and H is not 3-edge-colorable. There are no cubic graphs on at most 6 vertices which are not 3-edge-colorable. We have finally derived a contradiction. Graph G does not have a $(3k+1, k)$-edge-coloring, hence its circular chromatic index is greater than $3 + 1/k$. $\square$

Theorem 14. *Let G be a connected cubic graph with $4k+2$ vertices. If G does not have a 3-edge-coloring, the following holds:*

(i) If $k = 2$ then $\chi'_c(G) > 7/2$.

(ii) If $k = 3$ then $\chi'_c(G) \geq 17/5$.

(iii) If $k \geq 4$ then $\chi'_c(G) \geq 3 + \frac{1}{k-1}$.

Proof. By applying Lemma 13 we obtain $\chi'_c(G) > 3 + 1/k$. We have proved the part (i).

Let $k \geq 3$. By applying Lemma 11 we obtain $\chi'_c(G) \geq \frac{6k-1}{2k-1}$. We have proved the part (ii).

Let $k \geq 4$. Assume $\chi'_c(G) = \frac{6k-1}{2k-1}$. According to Theorem 8 we have a $(6k-1, 2k-1)$-edge-coloring such that every one of $6k-1$ colors $0, 1, 2, \ldots, 6k-2$ is used at least once. We have $6k+3$ edges in G. These two facts imply that four edges are colored by some colors already used. We have five possibilities how to write 4 as an unordered sum of positive integers: $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. First we exclude the possibilities containing a number greater than 1.

Assume that some color i is used at least three times (this corresponds to a number 2 or greater in the partitions above). Consider the matchings (sets of independent edges)

$$M_1 = c^{-1}(\{i-1, i, \ldots, i+2k-3\}) \qquad \text{and} \qquad M_2 = c^{-1}(\{i, i+1, \ldots, i+2k-2\}).$$

Both have to be perfect, because they have at least $2k+1$ edges. This means that the colors from $\{i-1, i+1, i+2, \ldots, i+2k-2\}$ are used exactly once (if they were used more times, we would have $2k+2$ or more edges in a matching which is impossible). This means that the matchings M_1 and M_2 differ in a single edge (colored by the color $i-1$ or $i+2k-2$ respectively). This is impossible for any simple graph.

There remains just one case to deal with; $1+1+1+1$. Assume that the four *busy* colors are used twice and all other colors are used once.

Imagine all colors $0, 1, \ldots, 6k-2$ in this order along a circle of length $6k-1$ with unit distances between consecutive colors. We have four busy colors, so there are two busy colors i, j such that the distance between them is at most $(6k-1)/4 < 2k-2$. Assume that $j \in \{i+1, i+2, \ldots, i+2k-3\}$. Consider the matchings

$$M_1 = c^{-1}(\{i-1, i, \ldots, i+2k-3\}) \qquad \text{and} \qquad M_2 = c^{-1}(\{i, i+1, \ldots, i+2k-2\}).$$

Both of them have at least $(2k-3)\cdot 1 + 2\cdot 2 = 2k+1$ edges, so they are perfect. Moreover, if color $i-1$ (or $i+2k-2$) is used more than once, then M_1 (or M_2) has more than $2k+1$ edges, which is impossible. Therefore M_1 and M_2 are perfect matchings, which differ in a single edge. This is a contradiction.

By the argument above we have proved that $\chi'_c(G) > \frac{6k-1}{2k-1} = 3 + \frac{2}{2k-1}$. The value $\chi'_c(G)$ is a fraction with denominator at most $2k+1$. By applying Lemma 12 we obtain $\chi'_c(G) \geq 3 + \frac{1}{k-1}$ as desired. $\square$

Theorem 15. *Let G be a cubic graph with $4k$ vertices. If G does not have a 3-edge-coloring, then $\chi'_c(G) \geq 3 + \frac{1}{k-1}$.*

Proof. Assume G has a $(3k+1, k)$-edge-coloring $c : E(G) \to \{0, 1, \dots, 3k\}$. Graph G is cubic, so it has $6k$ edges. Hence at least one color is used at most once. Let this color be 0. Consider the sets of edges

$$M_1 = c^{-1}(\{1, 2, \dots, k\}), \qquad M_2 = c^{-1}(\{k+1, k+2, \dots, 2k\}),$$

$$M_3 = c^{-1}(\{2k+1, 2k+2, \dots, 3k\}).$$

The set M_i is a matching in G, as any two edges e and f from M_i are not adjacent because of the inequality $|c(e) - c(f)| \geq k$. Moreover, the matchings M_1, M_2, and M_3 are pairwise disjoint. If two of them were perfect matchings, we would have a 3-edge-coloring of G. This would be a contradiction, because G is not 3-edge-colorable.

A perfect matching in G has size $2k$. Between M_1, M_2, and M_3 we have at most one matching with such cardinality, other consist of at most $2k-1$ edges. We now derive a contradiction.

$$\begin{aligned} 6k &= |E(G)| = |c^{-1}(\{0\}) \cup M_1 \cup M_2 \cup M_3| \leq 1 + |M_1| + |M_2| + |M_3| \\ &\leq 1 + 2k + 2k - 1 + 2k - 1 = 6k - 1. \end{aligned}$$

We have proved that G does not have $(3k+1, k)$-edge-coloring. Therefore the circular chromatic index of G is greater than $3 + 1/k$.

All cubic graphs with fewer than 10 vertices are 3-edge-colorable – cubic graph with fewer then 10 vertices cannot have a bridge and the smallest snark is the Petersen graph. Our graph G does not have a 3-edge-coloring, so $k \geq 3$. If $\chi'_c(G) = p/q$, then $q \leq 2k \leq 2k+1$ (Theorem 8). All conditions of Lemma 11 are satisfied. By applying this lemma we get that the circular chromatic index of G is at least $3 + \frac{2}{2k-1}$.

Assume $\chi'_c(G) = 3 + \frac{2}{2k-1}$. Then we have (Theorem 8) a $(6k-1, 2k-1)$-edge-coloring of G such that every one of the $6k-1$ colors $0, 1, \dots, 6k-2$ is used at least once. Graph G has $6k$ edges, so one color (say, 1) is used twice, any other color is used once. Consider the matchings

$$M_1 = c^{-1}(\{0, 1, 2, \dots, 2k-2\}) \qquad \text{and} \qquad M_2 = c^{-1}(\{1, 2, \dots, 2k-1\}).$$

Since both of them have $2k$ edges, they are both perfect; and they differ in a single edge. This is a contradiction.

So $\chi'_c(G)$ is a fraction greater than $3 + \frac{2}{2k-1}$ with denominator at most $2k$. For $k \geq 4$ we apply Lemma 12 and obtain $\chi'_c(G) \geq 3 + 1/(k-1)$. For $k = 3$ it is sufficient to verify that the least fraction with denominator at most 6 greater than 17/5 is 7/2. $\square$

Remarks. The established bounds could be improved a bit using the same methods. Later we will construct an infinite sequence of snarks on $4k+2$ vertices with circular chromatic index $3 + \frac{1}{ck}$ for some constant c. Therefore in some asymptotical sense the derived bound is tight. Note that we do not use the assumption that G is cubic in any essential way. The lower bounds can be generalized for Δ-regular graphs.

3.2 Known Values of the Circular Chromatic Index

The following result has been proved in [9], we give a much shorter proof and illustrate the use of our lower bound.

Theorem 16. *The circular chromatic index of the Petersen graph is* $11/3$.

Proof. A $(11,3)$-edge-coloring of the Petersen graph G is showed in Figure 3.2.

The Petersen graph G has $10 = 4 \cdot 2 + 2$ vertices. From Theorem 14 we know that $\chi'_c(G) > 3 + 1/2 = 7/2$. From Theorem 8 we have that for any (p,q)-edge-coloring of the Petersen graph $p \leq |E(G)| = 15$. Together with the inequality $p/q > 3$ it implies $q < 5$. The first fraction greater than $7/2$ with denominator $q < 5$ is $11/3$. Hence $\chi'_c(G) \geq 11/3$. $\square$

There are two infinite classes of snarks with known values of the circular chromatic index. Proofs of the upper bounds are easy – we only need to find a suitable coloring. Proofs of the lower bounds copy the idea of the original proof of the nonexistence of a 3-edge-coloring of these graphs. Both these infinite classes are constructed from small pieces joined together to form a circle. To determine possible colorings of the semiedges of the pieces is in the original proof used the Parity Lemma. While working with circular colorings, we do not have anything of that kind. The pieces used in construction are small enough and by a case analysis we can obtain enough information to find the lower bound and to prove this bound.

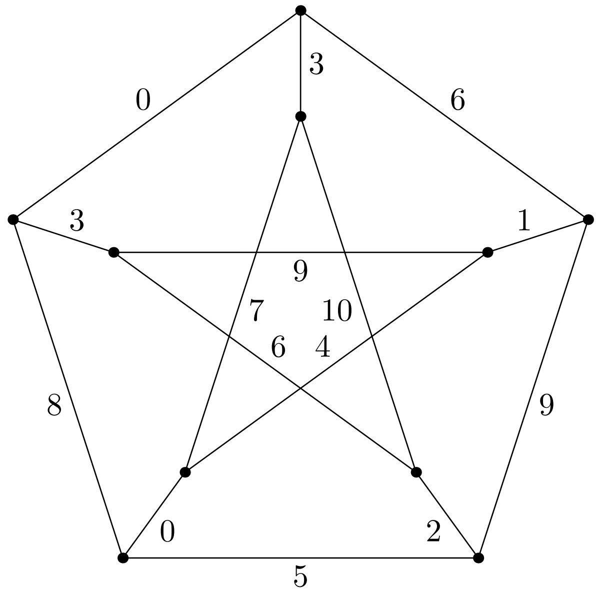

Figure 3.2: A (11,3)-edge-coloring of the Petersen graph.

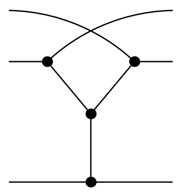

Figure 3.3: A piece of graph used in flower snarks construction.

Flower snarks

Flower snarks were constructed by Isaacs in [2]. We use the pieces showed in Figure 3.3, arrange them into a circle and join semiedges. From the Parity Lemma we know something about colors on semiedges of a piece. By considering a few cases we find out that if the number of used pieces is even, the resulting graph is 3-edge-colorable, and if the number is odd, then we have a snark F_{2k+1} consisting of $2k+1$ pieces. A complete proof can be found in [2].

The pieces used to construct a flower snark are very small. Hence it is possible to do a case analysis and capture some information about colors on semiedges. Some parity argument is used as in the original proof. The values (proved in [11]) are

$$\begin{aligned}
\chi'_c(F_3) &= 3.5, \\
\chi'_c(F_5) &= 3.4, \\
\chi'_c(F_{2k+1}) &= 3.\overline{3} \qquad \text{for } k \geq 3.
\end{aligned}$$

Goldberg snarks

The construction of the Goldberg snarks is based on the construction described by Loupekine. If we remove a path of length 2 from the Petersen graph, the resulting graph has five semiedges. We can choose four of them to form two pairs a, b and c, d in such a way that a, b has the same color if and only if c, d have different colors. This piece of a graph is sometimes called a negator. We join an odd number k of negators to form a circle; the resulting graph G_k is a snark.

Determining the circular chromatic index of Goldberg snarks is based on the idea of the original proof of the nonexistence of a 3-edge-coloring. The crucial step is to precisely capture the meaning of "the same color" and "different colors" in the terms of circular colorings. Moreover, the original proof uses the Parity Lemma, now we have to do without it – a place for a case analysis of negators, which are small enough in this case. The values determined in [12] are

$$\begin{aligned}
\chi'_c(G_3) &= 3 + \frac{1}{3}, \\
\chi'_c(G_{2k+1}) &= 3 + \frac{1}{4} \qquad \text{for } k \geq 2.
\end{aligned}$$

Chapter 4

Blanuša Snarks

Two of the first snarks known for a long time appeared in a paper of D. Blanuša. Isaacs showed in [2] that in fact they can be constructed by dot product from two copies of the Petersen graph. This construction can be generalized to form an infinite class of snarks (see [3]). These snarks are not too valuable from some point of view – they are not cyclically 5-edge connected. The Petersen graph is a member of this class, and it is known for its high circular chromatic index. Maybe this class contains other such snarks. Moreover, we want to know more about relations of the dot product and circular colorings. This class provides some examples.

Our aim is to determine the circular chromatic index of graphs in this class. They have rather regular structure – a pieces joined together to form a circle. As we have seen in the previous chapter, this is in fact the only type of larger graph, for which we are able to determine the circular chromatic index. First we give a description of this class and introduce some notation.

4.1 Generalization of Blanuša snarks

The basic piece A used in the construction of generalized Blanuša snarks is showed in Figure 4.1. We call it A-piece or A in the following. In any 3-edge-coloring of the A-piece the semiedges a and c are colored by the same color and b and d are colored by the same color. We prove this by the case analysis showed in Figure 4.2. Without loss of generality we can color the bottom edges by colors 0 and 1. The following two edges in the cycle of length 8 can be colored in four ways:

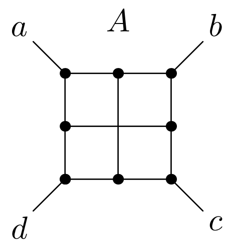

Figure 4.1: The basic piece used in the construction of Blanuša snarks.

(1) 2, 0; showed in Figure 4.2, part (1), with two neighbouring edges colored by 0, so this coloring is not possible,

(2) 2, 1; showed in Figure 4.2, part (2),

(3) 0, 2; showed in Figure 4.2, part (3), but after cyclic permutation of colors it is the same coloring as (2),

(4) 0, 1; showed in Figure 4.2, part (3).

So there are only two essentially different 3-edge-colorings of the A-piece: with a, b, c, d all of the same color or a, c of one color and b, d of another color. The property of the A-piece used to construct snarks is that a and c receive the same color in any 3-edge-coloring. We demonstrate it by constructing the Petersen graph from one A-piece (Figure 4.3). The semiedges a and c of A are joined to the vertex v, hence no 3-edge-coloring of the resulting graph is possible. It is easy to verify that this graph is isomorphic to the Petersen graph.

In the same way we can construct the first Blanuša snark using two A-pieces. The construction is shown in Figure 4.4. It is easy to see that again two of the edges

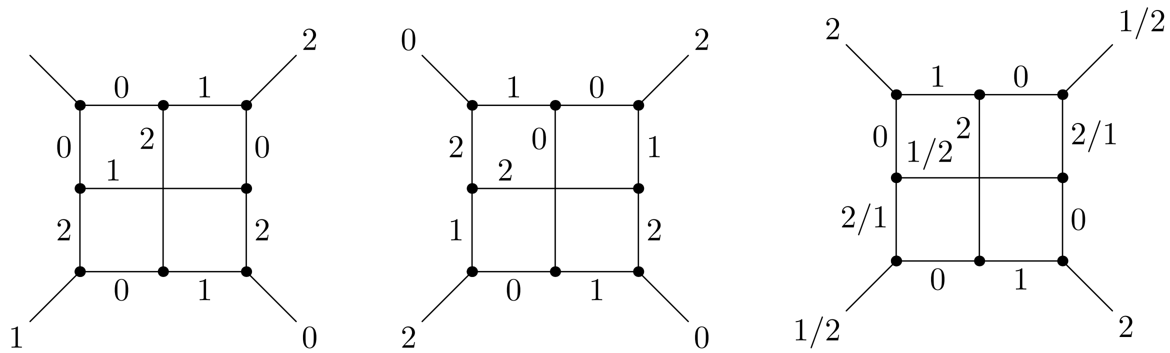

Figure 4.2: Possible 3-edge-colorings of the A piece.

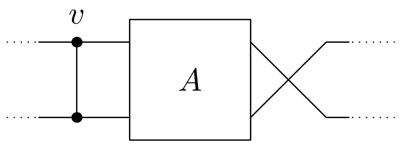

Figure 4.3: The Petersen graph constructed from the A-piece, the dashed edges wrap "around" the figure.

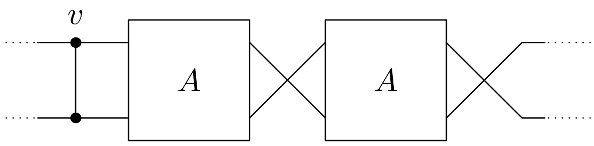

Figure 4.4: The first Blanuša snark constructed from two A-pieces.

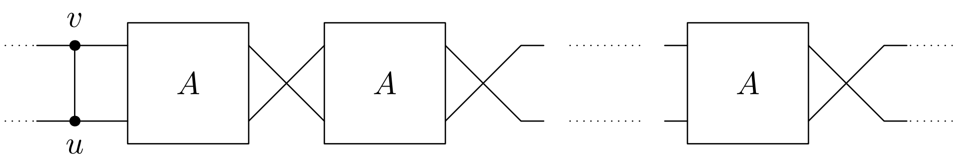

Figure 4.5: Generalized Blanuša snarks of the first kind.

incident with v have the same color in any 3-edge-coloring. Note that by removing two neighbouring vertices from the Petersen graph we obtain the A-piece. In the second copy of the Petersen graph we cut two edges to get four semiedges (as showed in Figure 4.3). Now we create the first Blanuša snark by a dot product from the prepared pieces. This process can be repeated as many times as we wish, and we get an infinite class of graphs showed in Figure 4.5. These graphs are snarks for the same reasons as the Petersen graph and the Blanuša snark are. As we have already said, they can be constructed by a dot product from a finite number of copies of the Petersen graph. The graph of this kind created from m A-pieces is denoted by B^1_m. The infinite class containing all these graphs is denoted by $\mathcal{B}^1$.

4.2 Circular Chromatic Index in the Class $\mathcal{B}^1$

Theorem 17. *The circular chromatic index of* B_m^1 *is*

$$\chi'_c(B_m^1) = 3 + \frac{2}{3m}.$$

The proof of this result is divided into three parts.

The lower bound

Why is B_m^1 a snark? We refine the original proof from the circular colorings viewpoint and derive a lower bound for the circular chromatic index of B_m^1. Look at Figure 4.5 and consider a 3-edge-coloring of B_m^1. The dashed edge incident with the vertex v is colored by some color, say, 0. This edge serves as an "input" for an A-piece, and as we have showed before, the edge on the "output" of that A-piece has the same color. So we have a sequence of edges of the same color beginning and ending in v. Call this path the *upper line*. Similarly is defined the *bottom line* passing through u. Existence of these lines implies that no 3-edge-coloring of B_m^1 exists. Now we are trying to find a $(3+\varepsilon)$-edge-coloring of this graph with ε as small as possible. Consider the upper line. After each A-piece the color on the upper line can be slightly changed – we have discussed this property of circular colorings in Chapter 3. What does it mean, "slightly"? We prove that the change of color made by one A-piece is at most 3ε.

We use the technique introduced in [12]. Consider a $(3+\varepsilon)$-edge-coloring of the A-piece. For any $a, b \in [0, r)$ the r-circular interval $[a, b]_r$ is defined as follows:

$$[a,b]_r = \begin{cases} [a,b] & \text{if } a \le b, \\ [a,r) \cup [0,b] & \text{if } a > b. \end{cases}$$

It is convenient to reduce a and b modulo r and use the same notation even if a or b are out of the interval $[0, r)$. Moreover, while r is fixed, we will usually write just $[a, b]$ for the r-circular interval $[a, b]_r$. In order for this technique to be succesful we need the $\varepsilon > 0$ to be small enough. Any color $x + y\varepsilon$ for integers x, y used as a boundary of the intervals in Figure 4.6 should be close enough to the color x. This is important in order to preserve the order of colors of the edges incident with a vertex. For example, if we have colors $1 + 2\varepsilon$ and $2 - 2\varepsilon$ then we need $1 + 2\varepsilon \le 2 - 2\varepsilon$, hence $\varepsilon < 1/4$. If we assume $0 < \varepsilon < 1/4$, all the necessary orders will be preserved. Therefore we will work under this assumption. The only exception are small graphs from the class $\mathcal{B}_1$; we deal with them later in this section.

Now back to our problem: we want to derive a lower bound. Suppose that the A-piece is colored in a $(3+\varepsilon)$-edge-coloring for $\varepsilon < 1/4$. Let the colors of the semiedges a, b, c, d be α, β, γ, δ in the indicated order. We prove that the total change of colors

$$D = |\alpha - \gamma| + |\beta - \delta|$$

satisfies the inequality $D \leq 3\varepsilon$. We may assume that $\alpha \leq \gamma$ and $\alpha = 0$. Hence $D = \gamma + |\beta - \delta|$.

The two edges adjacent with the semiedge a are colored by colors from $[1, 1+\varepsilon]$ and $[2, 2+\varepsilon]$ respectively. The A-piece is symmetric, hence we do not need to analyse the two cases. Then there are four possibilities to color the neighbouring edges. All are shown in Figure 4.6. In fact, these possibilities correspond to the three possible 3-edge-colorings of the A-piece (to see this, let $\varepsilon = 0$). The case in the bottom right corner is impossible. Note that the change of color on the upper line $|\gamma - \alpha|$ is at most 2ε, hence $\gamma \in [0, 2\varepsilon]$.

The colors of the edges incident with a vertex go along the circle of length $3+\varepsilon$ in some order. We analyse the three cases of a possible $(3+\varepsilon)$-edge-coloring. We will make use of the fact that the order of colors around any vertex is completely determined.

Consider the first case (upper left corner of Figure 4.6). The color γ is “almost 0”, and the color which follows γ in the mentioned order is “almost 1”. This color has to be at least $1+\gamma$. By repeating this argument we get that the next edge along the shortest path from c to d has color at least $2+\gamma$. Then $\delta \in [-2\varepsilon, \varepsilon]$ has to be in the distance at least 1 from $2+\gamma$. Therefore $\delta \in [\gamma - \varepsilon, \varepsilon]$. Similarly by repeating this argument along the shortest path from c to b we obtain that $\beta \in [\gamma - 2\varepsilon, 2\varepsilon]$.
If $\beta \geq \delta$, then $D = \gamma + \beta - \delta \leq \gamma + 2\varepsilon - (\gamma - \varepsilon) = 3\varepsilon$.
If $\beta < \delta$, then $D = \gamma + \delta - \beta \leq \gamma + \varepsilon - (\gamma - 2\varepsilon) = 3\varepsilon$.

In the second case (upper right corner) we have $\delta \in [2+\gamma-\varepsilon, 2+2\varepsilon]$ and the same bounds for β. Hence $|\beta - \delta| \leq 2 + 2\varepsilon - (2+\gamma-\varepsilon) = 3\varepsilon - \gamma$ and $D \leq 3\varepsilon$.

In the third case (bottom left corner) we have $1+\gamma-\varepsilon \leq \beta \leq 1+2\varepsilon$ and the same bounds for δ. Hence $|\beta - \delta| \leq 1 + 2\varepsilon - (1+\gamma-\varepsilon) = 3\varepsilon - \gamma$ and $D \leq 3\varepsilon$.

The total change of color for m A-pieces is has to be at least 2 (1 for the upper line and 1 for the bottom line), hence $2 \leq m \cdot 3\varepsilon$. Therefore $\varepsilon \geq \frac{2}{3m}$. The lower bound $\chi'_c(B^1_m) \geq 3 + \dfrac{2}{3m}$ is established.

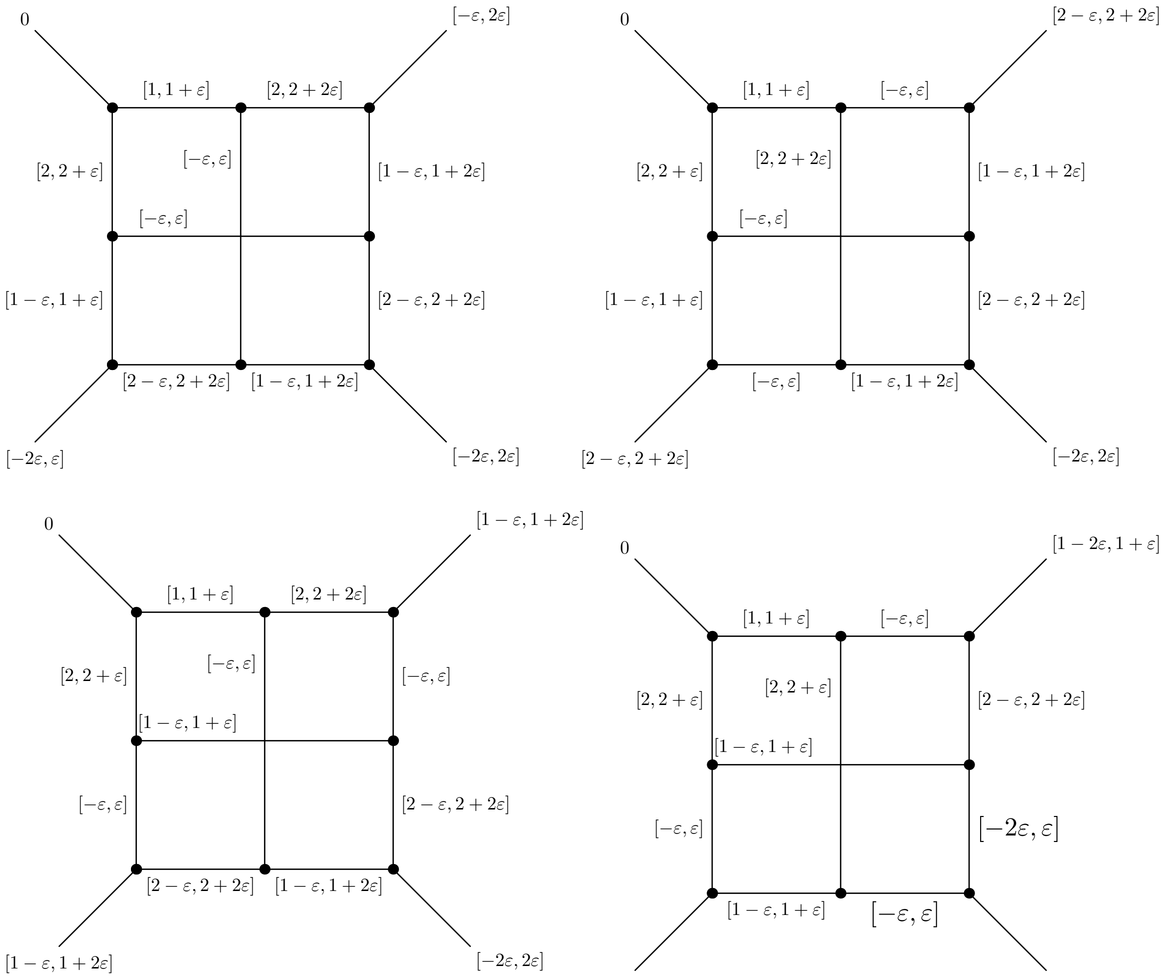

Figure 4.6: All possible $(3+\varepsilon)$-edge-colorings of the A-piece.

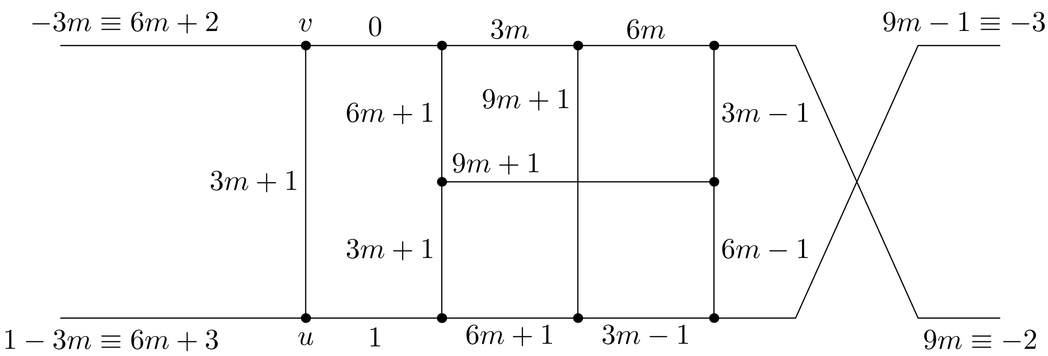

Figure 4.7: A $(9m+2, 3m)$-edge-coloring of the graph B^1_m.

The upper bound

Lemma 18. *The circular chromatic index of B_m^1 satisfies*

$$\chi_c'(B_m^1) \leq 3 + \frac{2}{3m}.$$

Proof. We prove the lemma by constructing a $(9m+2, 3m)$-edge-coloring of B_m^1. Look at Figure 4.7. The A-piece is colored in such a way that the color on the upper line decreases by 3 and the same holds for the bottom line. We repeat this coloring for all other A-pieces in B_m^1, decreasing colors by 3 after each repetition. It is clear that after m A-pieces the colors on both lines are decreased by $3m$. The edge uv is colored by color $3m+1$. It is easy to check that we have obtained a correct $(9m+2, 3m)$-edge-coloring of B_m^1. □

Small graphs from $\mathcal{B}_1$

For certain considerations above we needed $\varepsilon < 1/4$. Therefore we have to exclude all graphs with $\chi_c'(B_m^1) \geq 3 + 1/4$. In fact, for $m \geq 3$ the upper bound proved in Lemma 18 ensures $\chi_c'(B_m^1) \leq 3 + \frac{2}{3m} < 1/4$. We determine here the circular chromatic index of B_m^1 for $m = 1$ and $m = 2$.

Lemma 19. *We have*

$$\chi_c'(B_1^1) = 11/3, \qquad \chi_c'(B_2^1) = 10/3.$$

Proof. The first value was already proved in Theorem 16.

The graph B_2^1 has $18 = 4 \cdot 4 + 2$ vertices. From Theorem 14, part (iii) we have $\chi_c'(B_2^1) \geq 10/3$. Theorem 18 guarantees that there exists a $(20, 6)$-edge-coloring of B_2^1. □

Combining the established lower bound with the upper bound from Lemma 18 we get $\chi_c'(B_m^1) = 3 + \frac{2}{3m}$ for $m \geq 3$. Together with small cases analysed in Lemma 19 we have a complete proof of Theorem 17.

4.3 The Second Blanuša Snark

It is easy to find a $(7, 2)$-edge-coloring of the second Blanuša snark G. The lower bound from Theorem 14 is $\chi_c'(G) \geq 10/3$. This cannot be improved without considering the structure of G, as the first Blanuša snark B_1^1 would provide a counterexample for any such improvement. By a brute-force computer search we can exclude the possibility of $(17, 5)$-edge-coloring of G. Hence by Theorem 8 there are only two candidates for $\chi'_c(G)$: 24/7 and 7/2. The exact value remains as an open question. As the circular chromatic index of the first Blanuša snark is 10/3, we see that by performing a dot product on two copies of the Petersen graph we can obtain graphs with different values of the circular chromatic index.

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Abstrakt

Cirkulárne hranové r-farbenie grafu G je zobrazenie $c : E(G) \to [0, r)$ také, že pre každé dve susedné hrany e a f grafu G platí $1 \leq |c(e) - c(f)| \leq r - 1$. Circulárny chromatický index $\chi'_c(G)$ je infimum zo všetkých reálnych čísel r, pre ktoré má graf G cirkulárne hranové r-farbenie.

V práci stanovíme a dokážeme všeobecný dolný odhad cirkulárneho chromatického indexu snarku G. Tento odhad závisí len od počtu vrcholov grafu G a je asymptoticky tesný. Ďalej určíme cirkulárny chromatický index zovšeobecnených Blanušových snarkov. Index snarkov z tejto triedy môže nadobúdať nekonečne veľa hodnôt a môže byť ľubovoľne blízky číslu 3. Zovšeobecnené Blanušove snarky sú prvou známou triedou s touto vlastnosťou.

Kľúčové slová: snark, Blanušov snark, cirkulárny chromatický index