Univerzita Komenského, Bratislava
Fakulta matematiky fyziky a informatiky
Katedra informatiky

Real flows in graphs

Diplomová práca

Robert Lukoťka

Školiteľ: doc. RNDr. Martin Škoviera, PhD.
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Poďakovanie

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Abstract

A real flow on a graph is a flow with values in $\mathbb{R}$. A real nowhere-zero r-flow is a real flow φ with each edge satisfying the condition $1 \le |\varphi(e)| \le r-1$. The real flow number $\Phi_{\mathbb{R}}(G)$ of a graph G is the infimum of all reals r such that G has a real nowhere-zero r-flow. The purpose of this thesis is threefold.

First, we summarize and systematize the fundamental results of real flow theory. We give new proofs of several known results, in particular we present a new direct combinatorial proof of the existence of the minimal real nowhere-zero r-flow.

Second, we continue in the work of Z. Pan and X. Zhu who showed that for each rational number r between 2 and 5 there exist a graph with real flow number r [J. Graph Theory **49** (2003), 304-318]. We answer their question whether for each rational number $4 < r \le 5$ exists a snark with real flow number r by constructing an infinite family of snarks for each such r.

Finally, we obtain a lower bound on the real flow number of a snark of a given order and show that the Isaacs flower snarks attain this bound. As a consequence we show that the real flow number of the Isaacs snark I_{2k+1} is $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$, completing the upper bound of E. Steffen [J. Graph Theory **36** (2001), 24–34].

Key words. Real flows, real flow number, flow number, snark, Isaacs snarks.

Contents

Chapter 1

Introduction

The flow on a graph is a well-known and important concept in graph theory with many theoretical and practical applications. A k-flow on a graph G is an orientation of G and an assignment of non-negative integers smaller than k to the edges of G such that the sum of incoming values equals the sum of outgoing values. The concept of an A-flow, where A is an abelian group, is similar, but instead of integers we use elements of A and the addition in A. A nowhere-zero flow is one which does not use 0. The concepts of nowhere-zero A-flow and nowhere-zero k-flow are closely related – a graph has a nowhere zero A-flow, $|A| = k$ if and only if it has a nowhere-zero k-flow [20].

A systematic study of nowhere-zero flows was initiated by Tutte in [19] and [20] as a dual concept to vertex coloring. Among others, Tutte proposed the following important conjecture.

- 5-*Flow Conjecture:* Every bridgeless graph admits a nowhere-zero 5-flow.

Trivially, a graph with a bridge cannot have a nowhere-zero A-flow for any abelian group A (including the group of integers $\mathbb{Z}$). On the other hand, there is an immediate question whether there exist a bound n such that each graph G has a nowhere-zero n-flow. The 5-Flow Conjecture states 5 as this bound. The existence of such a general bound was first established independently by Kilpatrick [12] and Jaeger [11] who showed the bound to be 8. Later Seymour [16] proved that every bridgeless graph has a nowhere-zero 6-flow.

The 5-Flow Conjecture still resist all attempts of solving. During the time many different approaches have arisen, real flows being one of them. The fundamentals of real flow theory were set by Goddyn, Tarsi and Zhang in [6] by introducing the concept of a fractional flow and the star flow index. Again this notion was derived as the dual concept to (k, d)-*colorings* and the star chromatic number. Also the basic properties of these flows were acquired via this duality. One of the most important results says that these flows are in fact a refinement of integral flows. Later it has transpired that these notions can be introduced more naturally by employing rational or real flows instead of the previously used (k, d)-flows. A real nowhere-zero r-flow is then an $\mathbb{R}$-flow such that the values belong to the interval $\langle 1, r-1\rangle$. The real flow number of a graph G is the infimum of all reals r such that G has a real nowhere-zero r-flow. This infimum is known to be a minimum and also to be rational. These historical issues are one of the reasons why the terminology in this theory has become patchy. Therefore one of the objectives of this thesis is to unify the terminology and to establish the basic theorems by using the notions of real flow theory itself.

The question "*Which are the possible integral flow numbers of a graph?*" naturally suggests a similar question about real flows: *Which are the possible real flow numbers of a graph?* Pan and Zhu [15] showed that all rational numbers between 2 and 5 are possible real flow numbers. The graphs which they constructed for the rational numbers between 4 and 5 have several nice properties which led them to ask whether for each number greater than 4 and smaller than 5 there exists a snark with whose real flow number is exactly this number. Since the existence of a 4-edge-coloring in a cubic graph is equivalent to the existence of a nowhere-zero 4-flow, the validity of the 5-Flow Conjecture would imply that these are the only possible real flow numbers of snarks. The second objective of this thesis is to give the affirmative answer to this question.

Steffen [18] showed, among others, that the Isaacs snark I_{2k+1} has a real nowhere-zero $(4 + 1/k)$-flow. During our attempt to show this flow to be the best possible we have found out that there exists a general bound of the "size" of a graph having a given real flow number. For regular graphs the "size" is the number of vertices of a graph. This fact allowed us to give a lower bound for the real flow number of a snark with a given number of vertices. For Isaacs snarks the resulting lower bound coincides with Steffen's upper bound and consequently establishes the real flow number of the Isaacs snark I_{2k+1}. Reporting this research is the third main objective of this thesis. This direction may lead to further interesting results.

Chapter 2

Preliminaries

2.1 Graphs

There are variable definitions of graphs. Some allow parallel edges, loops, semiedges, etc., some not. In this work all graphs are finite, undirected, and may contain both parallel edges and loops. Occasionally we consider directed graphs as graphs endowed with an orientation.

To be more precise, a *graph* is an ordered quadruple $G = (D, V, I, L)$ where D is a set of *darts*, V is a nonempty set of *vertices*, which is required to be disjoint from D, I is a mapping of D onto V and L is a permutation of D, such that L^2 is the identity and $L(x) \neq x$. The vertex set of G and the dart set of G are often denoted by $V(G)$ and $D(G)$. We sometimes write x^{-1} instead of $L(x)$.

The mapping I assigns to each dart its *initial vertex*, and the permutation L interchanges a dart and its *reverse*. The *terminal vertex* of a dart x is the initial vertex of x^{-1}. The set of all darts having a given vertex u as their common initial vertex is denoted by $D(u)$. The cardinality of $D(u)$ is the *valency* of the vertex u. The orbits of L are called *edges*. Each dart determines uniquely its *underlying edge*. A set of edges of a graph G is denoted by $E(G)$. An edge e is *incident* to a vertex v, if $v \in I(e)$. The set of vertices incident to e is denoted by $V(e)$. A *loop* is an edge, which is incident only with one vertex. Two edges are *parallel* if they are incident with the same two vertices. All basic notions of graph theory can be easily use in this model. For definitions and more information about graphs we refer to [2]. We recall that a *circuit* is a connected subgraph whose all vertices have valency 2. The *girth* of a graph G is the length of the shortest circuit of G, denoted by $g(G)$.

Let $S \subseteq V(G)$. Then the *boundary* of S is the set of edges incident with one vertex for S and one vertex from $V(G) - S$ is denoted by $\delta_G S$. Let H be a subgraph of G. A *contraction* of G according to H, denoted by G/H, is a graph G', such that all components of H are contracted into a single vertex.

An *directed graph* is a graph G with an orientation O. The orientation O is a function from the set of edges to the set of darts such that $O(e) \in e$; $O(e)$ is called the *preferred dart*. The *initial* and the *terminal* vertex of an edge is an *initial* respectively *terminal* vertex of its preferred dart. A *circuit* in a directed graph is a circuit such that the terminal vertex of the an edge is also the initial vertex of another edge.

Let $S \subseteq V(G)$. The set of edges incoming to S (denoted S^-) is the set containing all edges from $V(G) - S$ to S. The set of edges outgoing from S (denoted S^+) is the set containing all edges from S to $V(G) - S$.

2.2 Flows in graphs

Let A be an abelian group written additively and let G be a graph. An *A-chain* on G is a function $\varphi : D(G) \to A$ satisfying the following condition.

(F1) $\varphi(x^{-1}) = -\varphi(x)$.

For a vertex v, let $\partial\varphi(v) = \sum_{x\in D(v)} \varphi(z)$. This value is the *outflow from v with respect to φ*. A vertex with a non-zero outflow is called *singular*. An A-chain φ is an *A-flow* if it has no singular vertices, that is, if the following property holds:

(F2) $\partial\varphi(v) = 0, \quad$ for each vertex $v \in V(G)$.

A flow is said to be *nowhere-zero* if $\varphi(z) \neq 0$ for each dart $z \in D(G)$. If $A = \mathbb{Z}$ and $|\varphi(x)| < k$ for each dart x, φ is also called a *k-chain*, *k-flow* and a *nowhere-zero* k-flow, respectively. A flow on a graph G induces a flow on G/H for every subgraph H of G by simply keeping the values on darts that have not been contracted. An *outflow* from H a subgraph of a graph G, $\partial\varphi(G)$ is the sum of outflows of all vertices of H.

From the definition it is easy to see that if a graph has a nowhere-zero k-flow, it also has a nowhere-zero k'-flow for any $k' \geq k$. It can easily be shown that every bridgeless graph has a nowhere-zero k-flow for some k. Therefore, for a bridgeless graph G, there exists the smallest integer k, such that G has a nowhere-zero k-flow. This value is called the *flow number* of G and is denoted $\Phi_{\mathbb{Z}}(G)$.

The following two theorems describe the relationship between integer k-flows and flows in abelian groups of order k.

Theorem 2.2.1. [20] *A graph G has a nowhere-zero k-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow.*

Theorem 2.2.2. [20] *Let A and B be two abelian groups of equal order. Graph G has a nowhere-zero A-flow if and only if it has a nowhere-zero B-flow.*

The flow number of a bridgeless graph is always at least 2. A fundamental question is which integers are the possible values for the flow number of a graph. It is easy to construct graphs with flow numbers 2, 3, 4 and 5, but graphs with other flow numbers are not known. This led Tutte to propose the following two conjectures.

Conjecture 2.2.1. *(5-Flow Conjecture)* [20] *Every bridgeless graph has a nowhere-zero 5-flow.*

Conjecture 2.2.2. *(3-Flow Conjecture)* [21] *Every graph without a 3-edge cut has a nowhere-zero 3-flow.*

The best approximations of these conjectures are theorems by Seymour [16] and Jaeger [10], respectively.

Theorem 2.2.3. *Every bridgeless graph has a nowhere-zero 6-flow.*

Theorem 2.2.4. *Every graph without edge 3-cuts has a nowhere-zero 4-flow.*

One of the possible approaches to these conjectures is o reduce the scope for counterexamples as much as possible. For example, it is known that it suffices to establish the 5-Flow Conjecture for cubic graphs [16] and the 3-Flow Conjecture for 5-regular graphs [18]. In this thesis we study flows on cubic graphs. In this case, there is an important relationship between edge colorings and 4-flows. [2]

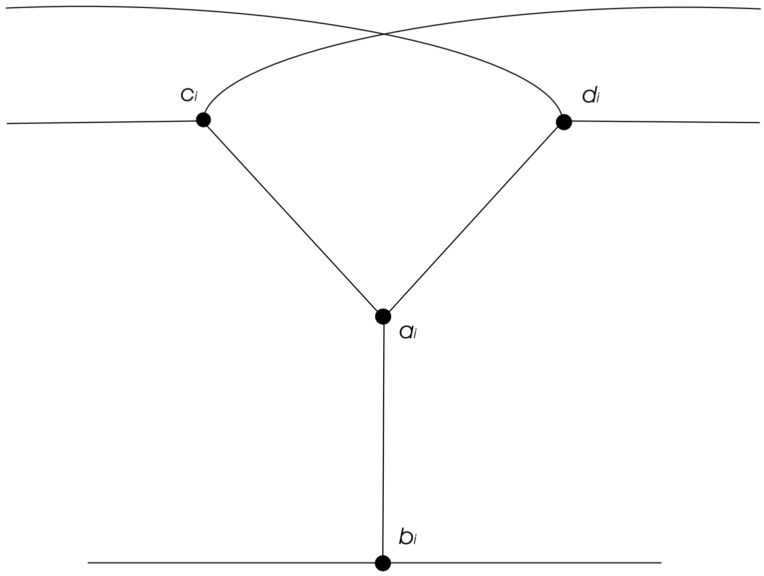

Figure 2.1: Basic segment of Isaacs snarks.

Theorem 2.2.5. [20] *For a cubic graph G the following statements are equivalent:*

1. *The graph G has a* 3*-edge coloring.*

2. *The graph G has a nowhere-zero* 4*-flow.*

3. *The graph G has a nowhere-zero* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$*-flow*

For more information about flows we refer the reader to [2]. For additional information about nowhere-zero flows we refer to [17] and [11].

2.3 A little about snarks

A *snark* is a 'non-trivial' cubic graph whose edges cannot be colored with 3 colors. Gardner [4] took the name 'snark' from Lewis Carroll's "Hunting of the Snarks" to mean a graph which is very difficult to find. The interest in these graphs grew as it was found out that several significant conjectures about graphs would have snarks as minimal counterexamples. Despite their simple definition and years of investigation very little is known about these graphs.

'Non-triviality' of snarks is usually defined as follows. A snark is usually assumed to be cyclically 4-edge-connected cubic graph with girth at least 5 and chromatic index 4. A graph is said to be *cyclically k-edge-connected* if deleting fewer than k edges does not disconnect the graph into components each containing a circuit. This constraint is brought to avoid cases where it would be trivial and non-interesting to construct a snark – for instance this would happen with the occurrence of a bridge in a snark as there is no cubic graph with a bridge that can be 3-edge-colored. Note that, by Theorem 2.2.5, the condition to have the chromatic index 4 is equivalent to the non-existence of a nowhere-zero 4-flow.

For a long time since the snarks were introduced, only a few snarks were known. The first infinite classes of snarks was constructed by Isaacs [8]. Isaacs also gave a construction which created an infinite family of snarks from non-snarks. These graphs are called *Isaacs*

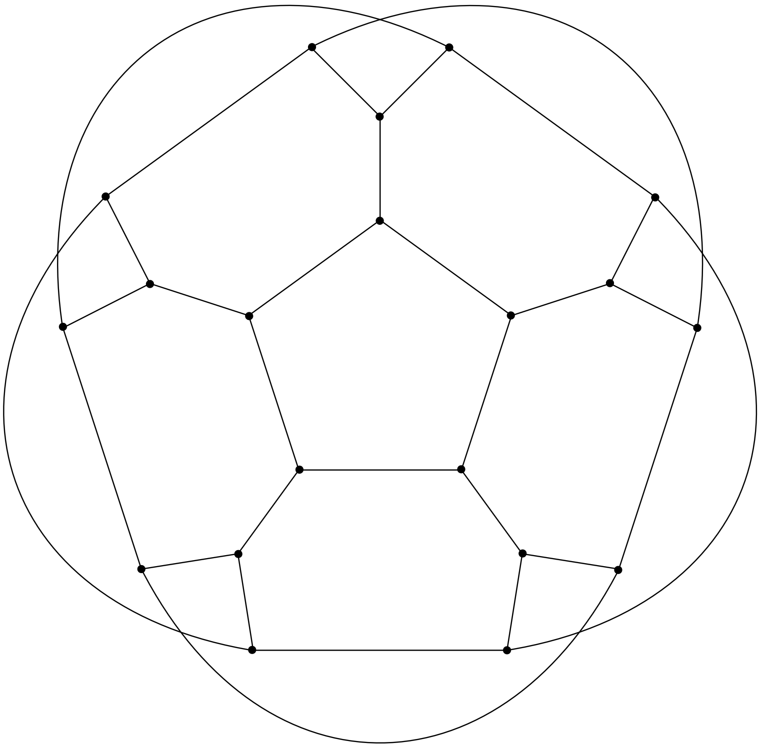

Figure 2.2: Isaacs snark I_5.

snarks or *flower snarks* [8]. So far this is the only known construction that produces snarks from non-snarks.

The Isaacs snark (Figure 2.2) I_{2k+1} is the graph with the vertex set $V(I_{2k+1}) = \{a_i, b_i, c_i, d_i \mid i = 0 \dots 2k\}$ and the edges

$$E(I_{2k+1}) = \{a_ib_i, a_ic_i, a_id_i, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} \mid i = 0, \dots, 2k\},$$

with indices reduced modulo $2k+1$ (Figure 2.1). Since then, many construction methods have been developed, but the class of Isaacs snarks is the only class of 6-edge-connected snarks. Because of their extraordinarily their are often used to test different hypotheses. Therefore their properties are intensively investigated ([3, 14, 5]).

Chapter 3

Real flow theory

3.1 Real flows and real flow number

Tutte's conjectures still resist attempts solving. Therefore many different approaches to them have arisen. The concept of real flows and real flow number was introduced in [6] as a (k, d)-flows and star flow index. The idea is to refine the concept of the flow number of a graph.

Definition 3.1.1. *Let G be a graph and let k, d be two positive integers $k \geq 2d$. A* (k, d)-flow *on G is a $\mathbb{Z}$-flow φ such that range of φ is in $\{\pm d, \pm(d+1), \cdots \pm (k-d)\}$. The* star flow number $\Phi^*(G)$*is the infimum of k/d over all (k, d) flows on G*

It is possible to take a more general approach. The relationship between these two definitions will be discussed later.

Real flow theory is a natural extension of integral nowhere-zero flow theory. Since the set of real numbers $\mathbb{R}$ under addition forms an abelian group, it is possible to consider $\mathbb{R}$-flows on graphs. By analogy to integer k-chains, k-flows and nowhere-zero k-flows, we define their real counterparts. Let $r \geq 2$ be a real number. A *real r-chain* φ is an $\mathbb{R}$-chain such that for every dart d, $|\varphi(d)| \leq r - 1$. A *real r-flow* is a real r-chain which is a flow. A *real nowhere-zero r-flow* is an r-flow satisfying $|\varphi(x)| \geq 1$. The *real flow number* $\Phi_{\mathbb{R}}(G)$ of a graph G is the infimum of the set of all real numbers r, such that G has a real nowhere-zero r-flow. Note that if we chose $|\varphi(x)| \neq 0$ instead of $|\varphi(x)| \geq 1$ (as by nowhere-zero flows in groups), we could divide all flow values to create a real nowhere-zero r-flow for any $r > 1$ on any bridgeless graph.

Having a flow r-flow φ on a graph G, for r a real number, we can consider the darts with positive flow value as preferred. Such an orientation is called the *positive orientation* of G with respect to φ. The flow through an edge e, $\varphi(e)$, is the flow through the preferred dart of the positive orientation of G with respect to φ.

Analogously we can define a rational *q-chain*, a rational *q-flow* and a rational *nowhere-zero q-flow* for q is a rational number. The *rational flow number*, $\Phi_{\mathbb{Q}}(G)$, of a graph G, is the infimum of all rational numbers q such that G has a rational nowhere-zero q-flow.

First we turn our attention to real flows. The first important result is that the infimum in the definition of real flow number can be replaced by a minimum.

Theorem 3.1.1. [6, 18] *Let G be a bridgeless graph with $\Phi_{\mathbb{R}}(G) = r + 1$. Then G has a real nowhere-zero $(r + 1)$-flow.*

Proof. Let $r = \inf\{s \in R;\ G$ has a real nowhere-zero s-flow$\} - 1$. Then for each positive integer n, G admits a real nowhere-zero $(r + 1/n + 1)$-flow, say φ'_n. Clearly, φ'_n can be viewed as an element of $\mathbb{R}^{D(G)}$. The sequence $(\varphi'_n)_{n=1}^{\infty}$ is clearly bounded and therefore it has a convergent subsequence $(\varphi_n)_{n=1}^{\infty}$. Let φ be limit of this sequence. Then the following hold:

- (F1) $\varphi(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} -\varphi_n(x^{-1}) = -\lim_{n\to\infty} \varphi_n(x^{-1}) = -\varphi(x^{-1})$,
- (F2) $\sum_{x\in D(v)} \varphi(x) = \sum_{x\in D(v)} \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \sum_{x\in D(v)} \varphi_n(x) = \lim_{n\to\infty} 0 = 0$.

From the definition of a limit we get $1 \le |\varphi(x)| \le r$. Therefore φ is a real nowhere-zero $(r+1)$-flow. ☐

A combinatorial proof of Theorem 3.1.1 will be given later.

Theorem 3.1.2. [6, 18] *Let G be a bridgeless graph. Then $\Phi_{\mathbb{R}}(G)$ is a rational number.*

Proof. Assume $\Phi_{\mathbb{R}}(G) = r+1$. Let φ be a real nowhere-zero $(r+1)$-flow such that the number of darts satisfying $\varphi(d) = r$ is minimal. Let us consider the positive orientation with respect to φ. Clearly there is an edge e carrying the value r. Let a and b be two vertices incident with e. We delete all edges carrying either the value 1 or r. The resulting graph G' is clearly disconnected – there is no path between a and b. If such a path P would exist, then we could add a sufficiently small flow value to the circuit Pe of G, in the direction opposite to the positive orientation of e. This would decrease the number of edges of G with $\varphi(d) = r$. Therefore G' consist of several components.

Let $G'' = G/G'$. The edges in G'' with $\varphi(e) = r$ cannot form a circuit. Otherwise we could subtract a sufficiently small flow value from this circuit and decrease the number of darts with $\varphi(e) = r$, which contradicts the choice of φ.

This means that there exists a vertex v of G'' where edges with $\varphi(d) = r$ only point to. Let k be the number of edges with $\varphi(d) = r$ flowing into this component, let l be the number of edges with $\varphi(d) = 1$ flowing into this component, and let m be the number of darts with $\varphi(d) = r$ flowing out of this component. Then

$$k.r + l.1 - m.1 = 0$$

Therefore r is rational. ☐

The next theorem shows that the real flow number of a graph may be viewed as a refinement of its flow number. This theorem also shows that our definition of the real flow number is equivalent to the definition of the star flow index.

Theorem 3.1.3. [18] *Let G be a graph, and let φ be a real nowhere-zero $(p/q+1)$-flow on G. Then there exists a rational nowhere-zero $(p/q+1)$-flow φ' on G such that for each $d \in D(G)$, $\varphi(d) = k/q$ for some $k \in \mathbb{Z}$.*

Proof. We proceed by induction on the number of darts such that $\varphi(d) \neq k/q$ for some $k \in \mathbb{Z}$. If this number is zero, the theorem is proved. Assume that this number is non-zero and let us take an edge e such that $|\varphi(e) - k/q|$ is minimal non-zero. We show that this edge must lie on some circuit containing only edges carrying the flow $\varphi(e') \neq k/q$.

If there was not such a circuit, the graph G' created from G by deleting edge e and edges carrying the value k/q would be disconnected. For the component X of G' that contains one of the incident vertices of e we would get

$$0 = q.\partial\varphi(X) = q. \sum_{d=(x,y),x\in X,y\notin X} \varphi(d) = \sum_{d=(x,y),x\in X,y\notin X} q.\varphi(d).$$

All but one of the numbers in the latter sum of are integers. Therefore we have a circuit containing only edges with $\varphi(d) \neq k/q$.

But $|\varphi(e)-k/q|$ is minimal, so we can add the value $|\varphi(e)-k/q|$ to the circuit according to the positive orientation with respect to φ. This decreases the number of darts such that $\varphi(d) \neq k/q$. $\square$

The previous theorems show that the occurrence of irrational numbers is not essential. Henceforth we can therefore assume that our flows only use rational values. Note that the proof of this theorem does not use previous results, and therefore can be used as a basis for a combinatorial proof of Theorem 3.1.1. We now establish two corollaries of Theorem 3.1.3 which show that the real flow number is a refinement of the flow number and the concept of a star flow index coincides with the rational flow number.

Corollary 3.1.4. [6, 18] *Let G be a bridgeless graph. Then*

$$\Phi_{\mathbb{Z}}(G) = \lceil \Phi_{\mathbb{R}}(G) \rceil.$$

Proof. By Theorem 3.1.1, graph G has a real nowhere-zero $\Phi_{\mathbb{R}}(G)$-flow. At the same time, the latter flow is a real nowhere-zero $\lceil \Phi_{\mathbb{Z}}(G) \rceil$-flow. By using Theorem 3.1.3, G also has a rational nowhere-zero $\lceil \Phi_{\mathbb{R}}(G) \rceil$-flow such that for each dart $\varphi(d) = k/1$. So φ is a nowhere-zero $\lceil \Phi_{\mathbb{R}}(G) \rceil$-flow. $\square$

We show that there is a one-to-one correspondence between (k, d)-flows and k/d-flows.

Corollary 3.1.5. *Let G be a bridgeless graph. Then $\Phi^*(G) = \Phi_{\mathbb{R}}(G) = \Phi_{\mathbb{Q}}(G)$.*

Proof. Let $\Phi^*(G) = k/d$, where k and d are two positive integers. It follows that G has a (k, n)-flow φ. We can simply obtain a rational (k/n)-flow φ' by setting $\varphi'(d) = \varphi(d)/n$ for each dart d. Conversely, if we have a real k/n-flow, using Theorem 3.1.3, we can obtain a rational nowhere-zero k/n-flow φ'' such that for all darts $\varphi''(d) = k/n$. Then we can create a (k, n)-flow φ''', by choosing $\varphi'(d) = \varphi(d).n$. $\square$

Now we are in position to give an entirely combinatorial proof of Theorem 3.1.1. The original proof of this result due to Goddyn [6] uses flow-coloring duality in matroids. We give a direct proof. We need the following simple number-theoretical result.

Lemma 3.1.6. *Let r be an irrational number, and let N be a positive integer. Then there exist positive integers p and q such that $(p-1)/q < r < p/q$ and all non-zero solutions of the diophantine equation $x.p/q + y.(p-1)/q + z = 0$ have one of $|x|$, $|y|$ and $|z|$ greater than N.*

Proof. We show that for each triple $(x, y, z) \neq (0, 0, 0)$ of integers, there is only a finite number of pairs $(p, q) \in \mathbb{N}^2$ satisfying the conditions of the lemma. The diophantine equation can be also rewritten as follows

$$(x + y)p + zq - y = 0 \tag{3.1}$$

From condition $(p-1)/q < r < p/q$ we obtain

$$\begin{aligned} p - qr &< 0 \\ p - qr - 1 &> 0 \end{aligned} \tag{3.2}$$

Let α be the angle between the lines $(x+y)p+zq-y = 0$ and $p-qr = 0$. Clearly $\alpha \neq 0$. Therefore the length of the line segment on the line (3.1) delimited by the area (3.2) is at most $1/\sin\alpha$ which is finite. Since the distance between two possible pairs (p, q) is at least 1, we have only a finite number of pairs (p, q) for each triple (x, y, z). Therefore there are only finite number of choices of q for each triple (x, y, z). But we have only N^3 "forbidden" triples (x, y, z), and therefore we have only finite "forbidden" choices of q. Now, if we choose an integer q which is not "forbidden", we can compute $p = \lceil rq \rceil$. This choice satisfies the conditions of lemma. □

Proof of Theorem 3.1.1. First we show that the infimum is rational. Suppose that $\Phi_{\mathbb{R}}(G) = r + 1$ is irrational. Using Lemma 3.1.6, there exist two integers p and q such that $(p - 1)/q + 1 < r < p/q + 1$ and all non-zero solutions of the diophantine equation $x.p/q + y.(p - 1)/q = z$ have one of $|x|$, $|y|$ and $|z|$ greater than $|D(G)|$. By our assumption, there is a real nowhere-zero $(p/q + 1)$-flow on G. By Theorem 3.1.3 there exists a rational-valued nowhere-zero p/q-flow on G with flow values k/q. From all such flows we take one for which the number of darts taking the value p/q is minimal. Let φ be such a flow. Let G' be the graph created by deleting this edges carrying flow values 1, $(p - 1)/q$ and p/q. The graph G' is clearly disconnected. The graph G/G' with orientation according to φ does not contain a directed circuit with edges having $\varphi(e) = p/q$. Therefore there exist a vertex where such darts with only point to. For that component we have that

$$0 = \partial\varphi(X) = x.p/q + y.(p - 1)/q + z.$$

But because for every non-zero solution of this diophantine at least one of $|x|$, $|y|$ and $|z|$ is greater than $d = |D(G)|$, the only possible solution is $(0, 0, 0)$. Therefore the flow φ is also a $((p - 1)/q + 1)$-flow, which contradicts the fact that $\Phi_{\mathbb{R}}(G) = r + 1$.

Now we show that $\Phi_{\mathbb{R}}(G)$ is rational. Assume that $\Phi_{\mathbb{R}} = p/q + 1$ and let $\epsilon = 1/(q.2^d)$. Clearly there exists a real nowhere-zero $(p/q + \epsilon + 1)$-flow on G. Theorem 3.1.3 guarantees a rational nowhere-zero $(p/q + \epsilon + 1)$-flow such that $\varphi(d) = k\epsilon$. We take an edge such that $|\varphi(e) - k/q|$ is minimal non-zero. This edge must lie on a circuit only containing edges carrying the flow $\varphi(e) \neq k/q$. So we can add $|\varphi(e) - k/q|$ in the positive orientation according to φ. This can be done until there are no edges with $p/q < \varphi(e) < (p + 1)/q$. But for every edge e in G with $\varphi(e) > p/q$, the change of its flow value is not large enough to increase $\varphi(e)$ to $(p+1)/q$, because the difference $|\varphi(e) - k/q|$ grows no more than twice in each step. So $\varphi(e) < (p + 1)/q$ for every edge and therefore the flow we have created is a nowhere-zero $(p/q + 1)$-flow. □

3.2 Modular flows

In this section we prove an analogue to Theorem 2.2.1. Let $r \geq 2$ be real number. Consider the quotient group $\mathbb{R}_r = \mathbb{R}/r\mathbb{Z}$ of all real numbers by the subgroup of all integral multiples of r. Every element of this group can be uniquely described by a real number $s \in \langle 0, r)$. The operation in $\mathbb{R}_r$ is the usual sum taken modulo r.

Chains or flows with values in $\mathbb{R}_r$ are usually called *modular* r-chains and *modular* r-flows respectively. A modular r-flow is *nowhere-zero* if $\varphi(x) \in \langle 1, r - 1\rangle$ for each dart x. As with k-flows and modular $\mathbb{Z}_k$-flows, the existence of a nowhere-zero r-flows is equivalent to the existence of a real nowhere-zero r-flows. As we shall see the proof is analogous.

Theorem 3.2.1. *Let G be a graph. Then G has a real nowhere-zero r-flow if and only if it has a modular nowhere-zero r-flow.*

Proof. If G has a real nowhere-zero r-flow, then its reduction modulo r is a nowhere-zero modular r-flow.

Conversely, let G have a modular nowhere-zero r-flow φ' on G. Choose a preferred orientation on G and for each edge replace the modular value carried by the preferred dart by the corresponding real number in the interval $\langle 1, r-1\rangle$. Assign the opposite value to the opposite dart. Let φ be the resulting r-chain. Clearly, for each vertex v, $\partial\varphi(v) \equiv 0 \pmod{r}$, so $\partial\varphi(v)$ is a multiple of r. Let V_1 be the set of vertices with positive outflow and V_2 the set of vertices with negative outflow. Suppose that we constructed φ in such way, that the outflow from V_1 is minimal. The outflow from V_1 is also clearly a multiple of r.

If the outflow from V_1 is zero, then $\partial_{V_1}\varphi = \partial_{V_2}\varphi$ and we have a nowhere-zero r-flow. If there is a vertex in V_1, there necessarily exist a vertex in V_2. There must be a path P from V_1 to V_2 with orientation according to φ. (otherwise there would be an edge cut separating V_1 and V_2 such that all cutting edges have positive orientation with respect to φ pointing to the component containing V_2 and the sum of flow in either component can not be zero.). So we can subtract r along P in the direction of positive orientation according to G. But the outflow from V_1 decreases then. Therefore φ is a rational nowhere-zero r-flow. $\square$

3.3 Rational flows and orientations of graphs

Given a graph G, a rational nowhere-zero flow on G uniquely determines the positive orientation of G with respect to the flow in question. This process can be reversed. Let us take an orientation O of a graph G. We can construct a nowhere-zero flow φ in such way that the preferred darts have positive flow value.

The definition of a flow on a graph can be also used in directed graphs, with demanding in addition that preferred darts have positive value. Such flows resp. chains are called *directed.*

If an graph G with an orientation O does not contain a set X into which edges only point to, we can construct a directed nowhere-zero flow on G. Analogously as for undirected graphs we can define *directed flow number* and *directed real flow number*, denoted by $\Phi_{\mathbb{R}}(G, O)$.

In the real-valued case the theorems and proofs are very similar to the undirected case. It is only necessary to replace the word “bridgeless" with the phrase “does not contain a set of vertices X into which darts only come in". This means, among others, that the infimum in the definition is minimum and is rational.

We prove an interesting relation between orientations of a graph and real flows.

Theorem 3.3.1. [6] *Let G be a bridgeless graph. Then G has a nowhere-zero real $(p/q+1)$-flow if and only if there exist an orientation O of G such that for each set X of vertices of G we have*

$$q/p \leq |X^+|/|X^-| \leq p/q.$$

Proof. Let φ be a real nowhere-zero $(p/q + 1)$-flow on G. Let us take the positive orientation with respect to φ. Then

$$q/p = 1/(p/q) \leq |X^+|/|X^-|) \leq (p/q)/1 = p/q.$$

Suppose that there exists an orientation O of G such that for every set of vertices X

$$q/p \leq |X^+|/|X^-| \leq p/q.$$

Suppose $\Phi_{\mathbb{R}}(G, O) > p/q + 1$. Let $r = \Phi_{\mathbb{R}}(G, O) - 1$ and let φ be a rational nowhere-zero $(r + 1)$-flow such that the number of darts with $\varphi(d) = r$ is minimum. Let us remove the edges carrying either the flow value 1 or r. Let us denote the sets of these edges E_1 and E_r, respectively. The new graph G' is not connected. Let $H = G/G'$. Because there cannot exist a circuit in H containing only the edges from E_r in the direction according to O and the edges from E_1 in the opposite direction, there exists a component X of G' into which the edges from E_r only come in and the edges from E_1 only go out. But

$$0 = \partial\varphi(X) = |X^+|.r - |X^-|.1.$$

Therefore

$$\Phi_{\mathbb{R}}(G, O) = r + 1 = \frac{|X^-|}{|X^+|} + 1 \leq p/q + 1.$$

Therefore G has a real nowhere-zero $(p/q + 1)$-flow. □

3.4 Real flows and balanced valuations

The previous result allows us to compute the real flow number using the orientation of edges. However, to compute the real flow number it is sufficient to do computations on vertices only. To show this we will need the concept of a balanced valuation on a graph which is due to Bondy [1] and Jaeger [9]. Let G be a graph. A *balanced valuation* of G is a mapping $w : V(G) \to \mathbb{R}$ such that for each subset $S \subseteq V(G)$, we have $|\sum_{v\in S} w(v)| \leq |\partial_G(S)|$. The theory of flows can be translated into the theory of balanced valuations by the following theorem of Jaeger [9].

Theorem 3.4.1. *A graph G has a rational nowhere-zero $(p/q + 1)$-flow $(0 < q < p)$ if and only if there is a balanced valuation w of G such that for each vertex $v \in V(G)$ there is an integer k_v for which $k_v \equiv d_G(v) \bmod 2$ and $w(v) = k_v(q + p)/(q - p)$.*

Proof. According to Theorem 3.3.1, the graph G has a rational nowhere-zero $(p/q+1)$-flow if and only if there exists an orientation O of the graph G such that for each $S \subseteq V(G)$ we have $q/p \leq |S^+|/|S^-| \leq p/q$. Let $k_v = \{v\}^+ - \{v\}^-$ (clearly $k_v \equiv d_G(v) \bmod 2$). Without loss of generality we can assume $|S^+| \geq |S^-|$. We get

$$\left|\sum_{v\in S} w(v)\right| = \left|\sum_{v\in S} (\ \{v\}^+ - \{v\}^-\ )\ \frac{q+p}{q-p}\right| = \left|\ (\ |S^+| - |S^-|\ )\ \frac{q+p}{q-p}\right| =$$
$$(\ |S^+| + |S^-|\ )\ \frac{|S^+|/|S^-| - 1}{|S^+|/|S^-| + 1}\ \frac{q+p}{q-p} \leq$$
$$|\partial_G(S)|\ \frac{p/q - 1}{p/q + 1}\ \frac{q+p}{q-p} = |\partial_G(S)|$$

Therefore w is a balanced valuation. Let now w' be a balanced valuation of G such that $w'(v) = k_v(q + p)/(q - p)$ and $k_v \equiv \partial_G(v) \bmod 2$. We can construct a new balanced valuation by setting $w(v) = 2k_v$ for each vertex v. Let $a_v = (k_v - |\partial_G(v)|)/2$. From the definition of a balanced valuation we get

$$\begin{array}{rcl} -|\partial_G(S)| \leq & \sum_{v\in S} w(v) & \leq |\partial_G(S)| \\ -|\partial_G(S)| \leq & \sum_{v\in S} (2a_v - |\partial_G(v)|) & \leq |\partial_G(S)| \end{array}$$

Let M be the set of edges with both endvertices in S. Then

$$\begin{array}{rcl} -|\partial_G(S)| \leq & \sum_{v\in S} 2a_v - 2|E(S)| - |\partial_G(S)| & \leq |\partial_G(S)| \\ 0 \leq & \sum_{v\in S} a_v - |E(S)| & \leq |\partial_G(S)| \end{array}$$

Therefore the following two conditions hold.

- For each $S \subseteq V(G)$ $\sum_{v \in S} a_v \geq |E(S)|$
- $\sum_{v \in G} a_v = |E(G)|$

Hakimi proved [7] that this two conditions are necessary and sufficient for G to have an orientation such that $|\{v\}^+| = a_v$. We can easily find out that $k_v = |\{v\}^+| - |\{v\}^-|$. Suppose that $|S^+| \geq |S^-|$. Since $|\sum_{v \in S} w(v)| \leq |\partial_G(S)|$, we get

$$(|S^+| + |S^-|)\frac{|S^+|/|S^-| - 1}{|S^+|/|S^-| + 1}\frac{q+p}{q-p} \leq |\partial_G(S)|\frac{p/q - 1}{p/q + 1}\frac{q+p}{q-p}$$

From this we can get that $q/p \leq |S^+|/|S^-| \leq p/q$ for every $S \subseteq V(G)$. $\square$

For a cubic graph, the only two possible values for k_v are $+1$ and -1. This splits the set of vertices of a graph G into two disjoint sets $V^+(G)$ and $V^-(G)$. Therefore we can rephrase the previous theorem for cubic graphs.

Theorem 3.4.2. *Let G be a graph. And let $0 < q < p$ be two integers. Graph G has a rational nowhere-zero $(p/q+1)$-flow if and only if there exists two disjoint sets $V_1 \subseteq V(G)$ and $V_2 \subseteq V(G)$ such that $V_1 \cup V_2 = V(G)$ satisfying For each subset $S \in V(G)$*

$$|\ |V_1 \cap S| - |V_2 \cap S|\ | \leq \frac{p+q}{p-q}\ |\partial_G(S)|.$$

3.5 The real flow number of some graphs

Theorem 3.1.2 shows, that the real flow number is always rational. It is also clear that all real flow numbers are at least 2. Due to Theorem 2.2.3 and Theorem 3.1.3, all real flow numbers are at most 6. If Tutte's 5-Flow Conjecture holds, all real flow numbers would be at most 5. In [22], it is proved that for any rational number $r \in \langle 2, 4\rangle$ there exist a planar graph with this real flow number. In [15] it is shown that there exist a graph with real flow number r also for $r \in (4, 5\rangle$. This construction will be shown latter.

However for regular graphs it is known that the set of possible values is not an interval due to following theorem of Steffen [18].

Theorem 3.5.1. *Let $k \geq 1$, for a $(2k+1)$-regular graph G the following statements are equivalent:*

- *Graph G is bipartite.*
- *The real flow number of G is $2 + 1/k$.*
- *Graph G admits a $(2 + \frac{r}{t})$-flow where $\frac{1}{k} \leq \frac{r}{t} \leq \frac{2}{2k-1}$.*

In particular for cubic graphs this means that there is no cubic graph with real flow number between 3 and 4.

Now we give some examples of real flow numbers of simple graphs. Steffen [18] has shown that for complete graph K_{2k+2} we have $\Phi_{\mathbb{R}}(K_{2k+2}) = 2 + 2/k$. It is also known by a result of Steffen [18], that the real flow number of the Petersen graph is 5. However there exist many classes of graphs whose real flow numbers are not known. One of these classes are the Isaacs snarks. The only result is the following theorem by Steffen [18]. In the last chapter we determine the real flow numbers of these snarks.

Theorem 3.5.2. *The graph I_{2k+1} has a nowhere-zero rational $(4 + 1/k)$-flow, moreover $\Phi_{\mathbb{R}}(I_{2k+1}) > 4$. For the real flow number of I_3 holds $\Phi_{\mathbb{R}}(I_3) = 5$.*

Chapter 4

Snarks with given real flow number

4.1 Two-terminal graphs

In this section we develop technique which will be useful for constructing snarks with given real flow number. Our construction consist of two parts. First we construct cyclically 4-edge-connected graphs with girth at least 5 which are not necessarily cubic. In the second step we split the vertices with bigger valency appropriately and get snarks.

From small graph we can construct a larger ones by many different techniques. However we will only use the simplest one – every smaller graph has only two vertices where it has to be joined with another one.

A *two-terminal* graph is a triple $(G; x, y)$ such that G is a graph and x and y are two distinct vertices of G, called *terminals*. If the two terminals are clear from context instead of $(G; x, y)$ we use G.

Let A be an abelian group and $(G; x, y)$ be a two-terminal graph. A *rooted A-flow* on is a nowhere-zero chain on G, such that the outflow from all vertices except x and y is 0. The *value* of a rooted A-flow is the outflow from x. A *rooted r-flow* on a two-terminal graph $(G; x, y)$ is a nowhere-zero modular r-chain such that for all non-terminal vertices the outflow from them is 0. The *value* of a rooted r-flow is the outflow from x. Set of values of all rooted r-flows on $(G; x, y)$ is called *r-transmissibility set* of $(G; x, y)$ and is denoted by $L_r(G; x, y)$.

The r-transmissibility set is symmetrical, that is with each element a it also contains $-a$. If a two-terminal graph has r-flow φ with value a, $-\varphi$ is a rooted r-flow with value $-a$. From the r-transmissibility set we can easily decide whether G has a rational nowhere-zero r-flow – a rooted r-flow with value 0 on $(G; x, y)$ is a modular nowhere-zero flow on G. This means that G admits a nowhere-zero r-flow if and only if $0 \in L_r(G; x, y)$. We can also create a new graph $G|_{x,y}$ by identifying the vertices x and y into one vertex. Such a graph admits a nowhere-zero r-flow if and only if $L_r(G; x, y) \neq \emptyset$.

The basic operations in construction of graphs are the parallel join and the series join. The *parallel join* $\|$ of two disjoint two-terminal graphs $(G; x, y)$ and $(G'; x', y')$, is the two-terminal graph $(G''; x'', y'')$ obtained from the union of these two graphs by contracting x and x' and y and y' into a single new vertices x'' and y'', respectively. The *series join* $\circ$ of tho disjoint two-terminal graphs $(G; x, y)$ and $(G'; x', y')$, is a two-terminal graph $(G''; x, y')$ obtained from the union of these two graph by contracting y and x' into a single new vertex y''. In both cases, from the r-transmissibility sets of $(G; x, y)$ and $(G'; x', y')$ we can easily find out the r-transmissibility set of newly constructed graph.

Let the *sum of two sets* A and B under the operation $+_r$, $A +_r B = \{a +_r b : a \in A, b \in B\}$.

Lemma 4.1.1. [15] *Let G and G' be two two-terminal graphs, then*

1. If $G'' = G \parallel G'$ then
$$L_r(G'') = L_r(G) \cap L_r(G').$$

2. If $G'' = G \circ G'$ then
$$L_r(G'') = L_r(G) +_r L_r(G').$$

This theorem allows us to construct a large variety of networks with known r-transmissibility set. Trivially, the r-transmissibility set ($r \geq 2$) of a two-terminal graph containing only one edge between the terminals is $L_r(G) = \langle 1, -1 \rangle$. Let P' be a two-terminal graph created from the Petersen graph by choosing two adjacent vertices as terminals and removing the edge between them. Because the real flow number of P is 5 [18] and the real flow number od P' is 4 [15] we can easily find out about the r-transmissibility set of P' [15]:

$$L_r(P'; x, y) = \begin{cases} \emptyset & 2 \leq r < 4 \\ \langle -(r-4), r-4 \rangle & 4 \leq r < 5 \\ \langle 0, r) & 5 \leq r \end{cases}$$

Now we can easily compute the r-transmissibility set of P_k a graph created with the series join of k copies of P' [15].

$$L_r(P_k; x, y) = \begin{cases} \emptyset & 2 \leq r < 4 \\ \langle -k(r-4), k(r-4) \rangle & 4 \leq r < 4 + 1/k \\ \langle 0, r) & 4 + 1/k \leq r \end{cases}$$

4.2 Graphs with given real flow number

In [15] Pan and Zhu presented a the construction of graphs with given real flow number. In this section we will follow this construction. A graph G is *feasible* if the valency of every vertex is at least 3, G is cyclically 4-edge-connected and has girth at least 5. A two-terminal graph $(G; x, y)$ is *feasible* if G is feasible and distance between terminals is at least 4; if we allow terminals to have valency (at least) 2, then $(G; x, y)$ is said to be *weakly feasible*. We prove that the graphs we get by Pan's and Zhu's construction are feasible. We start with the basic elements of construction.

Lemma 4.2.1. *The graph P' is weakly feasible.*

Proof. It is well known that P cyclically 5-edge-connected. This implies P' is cyclically 4-edge-connected. The other conditions are obvious. □

Lemma 4.2.2. *For $k \geq 2$, P_k is feasible.*

Proof. Since P' is cyclically 4-edge-connected with valency of terminals at least 2, every possible edge 3-cut must lie in different parts of P_k. But then one of parts must be cut by one edge. So P_k is cyclically 4-edge-connected. Let K be a circuit with at most 4 vertices. Since the distance between x and y is 4, the circuit K cannot lie in more then a single copy of P'. Since K is in one copy of P' it must have at least 5 vertices, because $g(P') = 5$. Moreover P_k has a circuit containing 5 edges. So $g(P_k) = 5$. Also $d(x, y) = 4$ and all the vertices have valency at least 3, so P_k is for $k \geq 2$ feasible. □

The next lemma will be used later. It shows that graphs remain feasible during the later presented construction

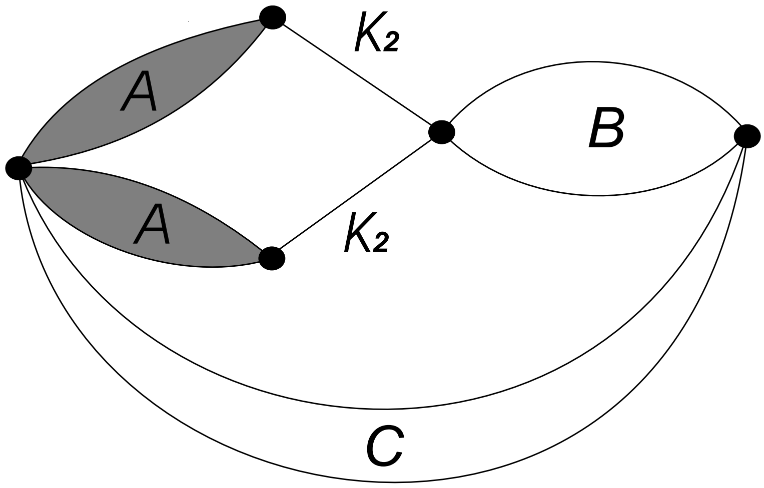

Figure 4.1: Graph G from Lemma 4.2.3.

Lemma 4.2.3. *Let A be a feasible two-terminal graph, and B, C two weakly feasible two-terminal graphs. Then the two-terminal graph*

$$G = (((A \circ K_2) \parallel (A \circ K_2)) \circ B) \parallel C$$

is feasible (Figure 4.1).

Proof. First it is easy to see that $d(x, y) \geq 4$ and $g(G) = 5$. Moreover G has no edge 2-cut. Any edge 3-cut separates only vertices inside A, B or C. But all these graphs are cyclically 4-edge-connected, and thus every edge 3-cut separates only one vertex. $\square$

To construct graphs with given real flow number we need an ordering of fractions. Let $r = p/q, p > 1, q > 1, (p, q) = 1$. Then there exist two unique integers $0 < a < p$ and $0 < b < q$ such that $pb - aq = 1$. The number a/b is called the *lower parent* of r ($\Lambda(r)$). The number $a'/b' = (p - a)/(q - a)$ is called the *upper parent* of r ($\Upsilon(r)$).

These are the basic properties of Λ and Υ.

Lemma 4.2.4. *Let $r = p/q$, $p, q > 0$, $\Lambda(r) = a/b$, $\Upsilon(r) = a'/b'$. Then*

$$\begin{aligned} \Lambda(r) < \ r \ &< \Upsilon(r) \\ \Lambda(\Upsilon(r)) &\leq \Lambda(r) \\ \Upsilon(\Lambda(r)) &\geq \Upsilon(r) \\ \Upsilon(r) &\leq (a+1)/b \\ \textit{For } r > 4 : \Upsilon(r) &\leq (a-4)/(b-1) \end{aligned}$$

Now we can proceed to the construction itself. The critical part is the proof of following lemmas.

Lemma 4.2.5. *For each $4 < p/q < 5$, $(p, q) = 1$, $a/b = \Lambda(p/q)$, $a'/b' = \Upsilon(p/q)$ there exist a feasible two-terminal graph G, such that for $a/b \leq r < a'/b'$*

$$L_r(G) = \langle -(qr - p + 1), qr - p + 1 \rangle .$$

For $r < a/b$

$$L_r(G) = \emptyset.$$

Proof. We will use induction on the denominator. If $\Lambda(p/q) = 4/1$, then $p/q = 4 + 1/k$. $\Upsilon(p/q) = 4 + 1/k - 1$. Let $G = P_k$. The graph G satisfies the conditions of the theorem.

Assume now that $\Lambda(p/q) > 4$, By induction hypothesis, there is a feasible two-terminal graph G', such that for every $\Lambda(a/b) \le r < \Upsilon(a/b)$

$$L_r(G') = \langle -(br - a + 1), br - a + 1 \rangle .$$

From Lemma 4.2.4, $\Lambda(a/b) \le a/b$, $a'/b' \le \Upsilon(a/b)$ and $\Lambda(a/b) \le a/b \le r \le a'/b' \le \Upsilon(a/b)$, so for $a/b \le r < a'/b'$ $L_r(G') = \langle -(br - a + 1), br - a + 1 \rangle$.

Let $Q = G' \circ K_2$. By Lemma 4.1.1,

$$L_r(Q) = L_r(G') \cap L_r(K_2) = \langle -(br - a + 1), br - a + 1 \rangle \cap \langle 1, -1 \rangle .$$

If $br - a + 1 < 1$, which is equivalent to $r < a/b$, we have that $L_r(Q) = \emptyset$. This will be true for all graphs created from this graph by performing join to the other graphs. From now on we suppose $a/b \le r < a'/b'$. Else

$$L_r(Q) = \langle 1, 1 + br - a \rangle \cup \langle -1 - (br - a), -1 \rangle .$$

Let $R = Q \parallel Q$. Then

$$\begin{aligned} L_r(R) &= \langle 1, 1 + br - a \rangle \cup \langle -1 - (br - a), -1 \rangle \\ &\quad + \langle 1, 1 + br - a \rangle \cup \langle -1 - (br - a), -1 \rangle \\ &= \langle -(br - a), br - a \rangle \cup \langle 2, -(2a - 2 - (2b - 1)r) \rangle \\ &\quad \cup \langle 2a - 2 - (2b - 1)r, -2 \rangle . \end{aligned}$$

Let now $G^* = R \circ P'$. We have $r - 4 < 2$. From Lemma 4.2.4, we obtain $r < a'/b' \le (a + 1)/b$. Using $r - 4 < 2a - 2 - (2b - 1)r$ from Lemma 4.2.4, we get $r - 4 > br - a$ so

$$L_r(G^*) = \langle -(br - a), br - a \rangle .$$

If $\Upsilon(p/q) = 5$, then $q = b + 1$ and $p = a + 5$. Let $G = G^* \parallel P'$. Then

$$\begin{aligned} L_r(G) &= L_r(G^*) + L_r(H) \\ &= \langle -(br - a), br - a \rangle + \langle -(r - 4), r - 4 \rangle \\ &= \langle -((b + 1)r - (a + 5) + 1), (b + 1)r - (a + 5) + 1 \rangle \\ &= \langle -(qr - p + 1), qr - p + 1 \rangle . \end{aligned}$$

Moreover, by Lemma 4.2.3, G is feasible.

If $\Upsilon(p/q) < 5$, by induction hypothesis, there is a feasible two-terminal graph G'', such that for $\Lambda(a'/b') \le r < \Upsilon(a'/b')$

$$L_r(G'') = \langle -(b'r - a' + 1), b'r - a' + 1 \rangle .$$

Let $G = G^* \parallel G''$. Since by Lemma 4.2.4, $\Lambda(a'/b') \le a/b$ and $a'/b' \le \Upsilon(a'/b')$

$$\begin{aligned} L_r(G) &= L_r(G^*) + L_r(G'') \\ &= \langle -(br - a), br - a \rangle + \langle -(b'r - a + 1), b'r - a + 1 \rangle \\ &= \langle -((b + b')r - (a + a') + 1), (b + b')r - (a + a') + 1 \rangle \\ &= \langle -(pr - q + 1), pr - q + 1 \rangle . \end{aligned}$$

Moreover by Lemma 4.2.3, the graph G is feasible. $\square$

The proof of the next theorem is now easy.

Theorem 4.2.6. *For every rational number $r = p/q$, $4 < r < 5$, there is 3-edge-connected, cyclically 4-edge-connected graph with girth 5, such that $\Phi_{\mathbb{R}}(G) = p/q$.*

Proof. By Lemma 4.2.5 there is a feasible two-terminal graph such that

$$L_r(G) = \langle -(qr - p + 1), qr - p + 1 \rangle$$

Let $(G'; x, y) = G \circ K_2$.

$$L_r(G') = L_r(G) \cap L_r(K_2) = \langle -(qr - p + 1), qr - p + 1 \rangle \cap \langle 1, -1 \rangle$$

which is empty if and only if $r \geq p/q$. Let $G'' = G'|_{x,y}$. Then $\Phi_{\mathbb{R}}(G'') = p/q$. Since G is feasible, G'' is 3-edge-connected, cyclically 4-edge-connected and with girth 5. □

4.3 Snarks with given real flow number

In this section we show that for each rational number p/q such that $4 < p/q \leq 5$ there exist a snark G with $\Phi_{\mathbb{R}}(G) = p/q$. To proof this we will need the notion of dipole. A *dipole* is a graph with two chosen distinct vertices as the *connectors*. The edges incident to this vertices are called the *dangling edges*.

Let D_1 and D_2 be two dipoles and v_1 and v_2 one of the connectors of D_1 and D_2, respectively, such that the valency of v_1 is the same as the valency of v_2. The *join* of two dipoles D_1 and D_2 is a dipole created by deleting v_1 and v_2 and joining edges that were incident to v_1 with the edges that ware incident to v_2. Note that the resulting graph is also dipole.

First we give a simple lemma.

Lemma 4.3.1. *Let G be a graph and let X be any subset of vertices of G. Then $\Phi_{\mathbb{R}}(G|_X) \leq \Phi_{\mathbb{R}}(G)$.*

Proof. Since every real nowhere-zero r-flow on G induces a real nowhere-zero r-flow on $G|_X$, the inequality $\Phi_{\mathbb{R}}(G|_X) \leq \Phi_{\mathbb{R}}(G)$ follows immediately. □

The critical part of the proof is this:

Lemma 4.3.2. *Let G be a feasible graph with $4 < \Phi_{\mathbb{R}}(G) < 5$. Then there exist a feasible cubic graph with the same real flow number.*

Proof. Let $\Phi_{\mathbb{R}}(G) = r$ and φ be a real nowhere-zero r-flow on G. To prove the lemma we will substitute each vertex v of valency greater than 3 by a cubic graph G_v with dangling edges (corresponding to the edges of G incident with v) in such a way that the resulting cubic graph is feasible and has an r-flow on G. From Lemma 4.3.1, we deduce that the real flow number of the new graph is the same as the real flow number of G, that is, r.

Consider the positive orientation of G with respect to φ. Let v be any vertex of valency greater than 3. The set of edges incident with v splits into the set $E^+(v)$ of edges directed from v and the set $E^-(v)$ of edges directed to v.

We describe two operations.

(1) Let e be a directed edge carrying a flow value $\varphi(e)$ such that $2 < \varphi(e)(< 4)$. Attach a dipole with a single input edge e and two output edges e_1 and e_2 carrying values $\varphi(e_1) \geq 1$ and $\varphi(e_2) \geq 1$ such that $\varphi(e_1) + \varphi(e_2) = \varphi(e)$ to e.

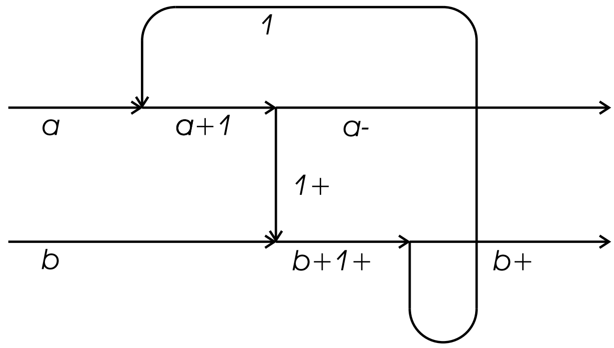

Figure 4.2: Simplified transferring construction.

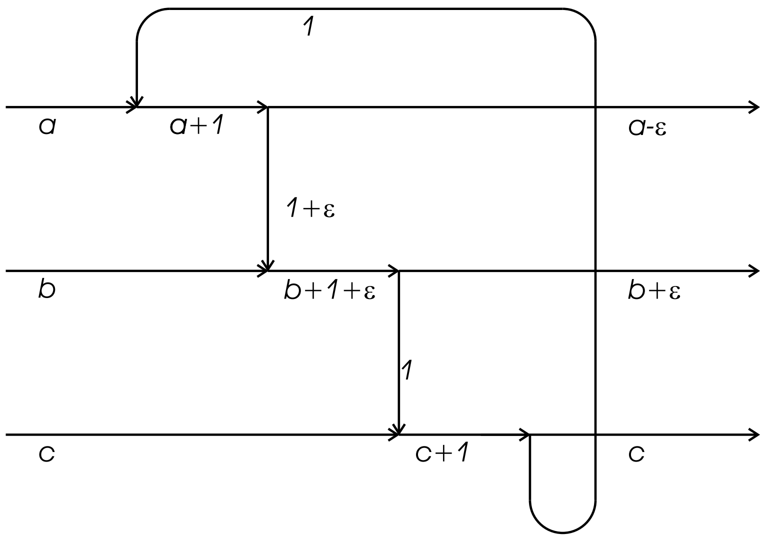

Figure 4.3: Simplified transferring operation for three edges.

(2) Let e and f be two directed edges carrying the flow values $\varphi(e)$ and $\varphi(f)$ such that $\varphi(e) < 2$ and $\varphi(f) < 2$. Let $\varepsilon > 0$ be a real number such that $\varepsilon \leq r - 4$ and $\varphi(e) \geq 1+\varepsilon$. We attach the dipole shown in Figure 4.2 with flow values as indicated to e and f.

Now let us remove the vertex v from G and retain the edges of $E^-(v)$ and $E^+(v)$ as dangling edges. Apply the operations (1) and (2) to $E^-(v)$ repeatedly to produce a dipole D^- whose input is the set $E^-(v)$ and the output is a set of edges such that all of them carry the flow value 1 except the last one which carries a value greater than 1 and smaller than 2. It is easy to see that this is always possible. We apply a similar procedure to $E^+(v)$, but in the reverse direction, to obtain a dipole D^+. By the flow conservation property, the number of output edges of D^- equals the number of input edges of D^+, so we can connect an edge of D^- with an edge of D^+ which carries the same value to obtain a new graph G'. The latter graph admits a nowhere-zero r-flow and has fewer vertices of valency greater than 3.

However the new graph is not feasible – it may contain short circuits and need not be cyclically 4-edge-connected. To avoid creating short circuits we employ a bit more complicated version of the operation (2) described above. We can easily generalize the dipole in Figure 4.2 to a transferring dipole which involves more than two edges (Figure 4.3). Moreover we can avoid forming short circuits (Figure 4.4) by inserting a circuit

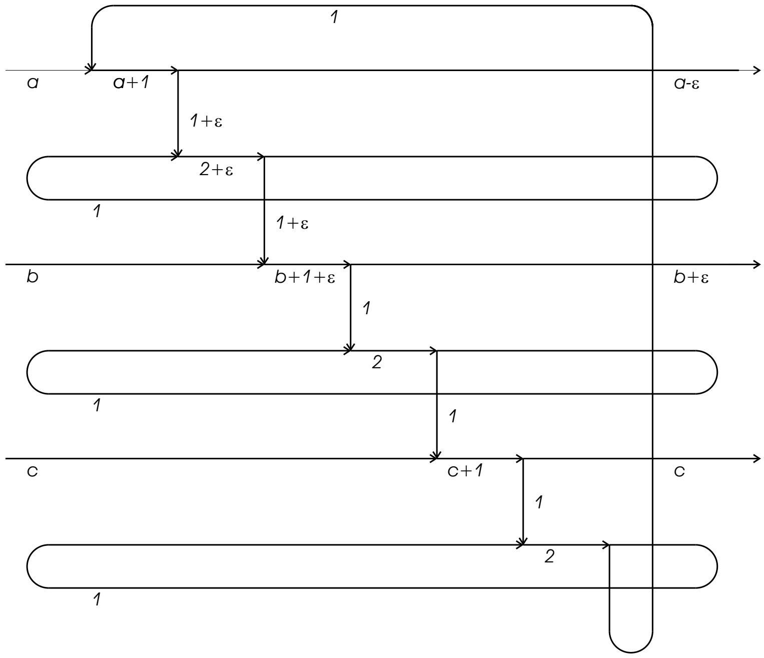

Figure 4.4: Transferring operation for three edges with circuits between these edges.

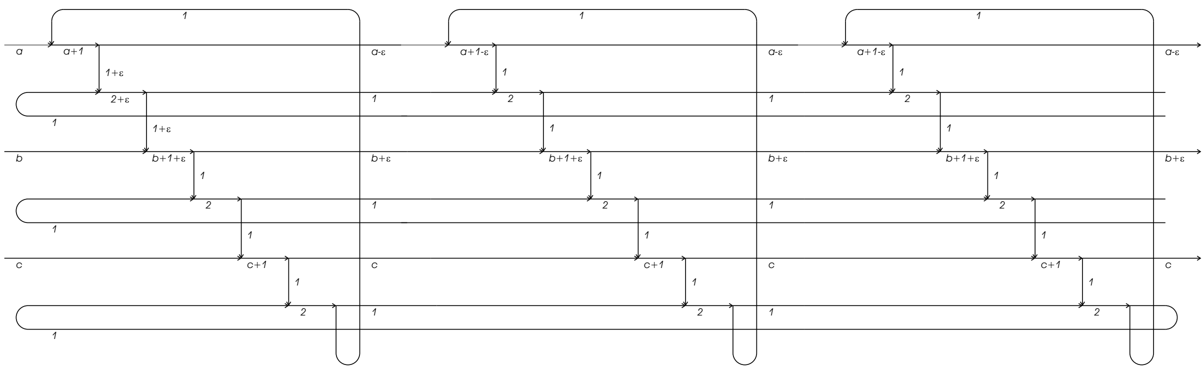

Figure 4.5: Transferring construction for three edges.

between any two edges and by using the more general construction three times with the same inserted circuits. The resulting multipole for three edges is shown in 4.5. Now instead of joining with the dipole in Figure 4.2 on edges e and f we will join the dipole in Figure 4.5 on all edges of E^- or E^+, respectively.

To retain cyclic connectivity we must join all edges somehow. This can be done simple, on the last step of construction when joining edges instead of joining the edges of E^- and E^+, we join both sets with dipole in Figure 4.5.

The resulting graph constructed this way is feasible and also we constructed a $\Phi_{\mathbb{R}}(G)$-flow on it. Using this construction on all vertices with valency at least 4 we get a cubic graph. $\square$

Theorem 4.3.3. *For any rational number $4 < r \leq 5$, there exists a snark G with $\Phi_{\mathbb{R}}(G) = r$.*

Proof. Because a feasible cubic graph is a snark, for $4 < r < 5$ we can use Theorem 4.2.6 and Lemma 4.3.2. If $r = 5$ let $G = P$. Petersen graph is a snark and $\Phi_{\mathbb{R}}(P) = 5$. $\square$

Note that the graphs constructed in Section 4.2 have always a vertex of valency at least 4. On these vertices we can use the transferring construction in infinitely many ways. Máčajová and Raspaud [13] showed that there exists infinitely many graphs with real flow number 5. Therefore we have following theorem.

Theorem 4.3.4. *For any rational number $4 < r \leq 5$, there exist infinitely many snarks G with real flow number r.*

Chapter 5

Lower bound on the rational flow number of a snark

5.1 Lower bound for the size of a graph with given real flow number

Let G be a graph. Let $\Phi_{\mathbb{R}}(G) = p/q + 1$ where p and q are two relatively prime positive integers. In this chapter we will try to decide how small actually such a graph can be. From Theorem 3.3.1 we can obtain following result.

Theorem 5.1.1. *Let G be a graph such that $\Phi_{\mathbb{R}}(G) = p/q + 1$ where p and q are two relatively prime positive integers. Then there exists a subset $S \subseteq V(G)$ such that both subgraphs of G induced by S and $V(S) - S$ are connected and*

$$\delta_G(S) \geq p + q.$$

Proof. Theorem 3.3.1 shows that there exist an orientation O and a subset S of vertices of G such that

$$\frac{|S^+|}{|S^-|} = \frac{p}{q} \tag{5.1}$$

Moreover, we can assume that both the graph $G[S]$ induced by S and the graph $G[\bar{S}]$ induced by $V(G) - S$ are connected. Indeed, if $G[S]$ was not connected, one of its components $G[T]$, where $T \subseteq S$, would have to satisfy (5.1). Now $G[T]$ is connected. If $G[\bar{T}]$ was disconnected, we could take a component $G[X]$ of $G[\bar{T}]$ induced by a subset $X \subseteq \bar{T}$. Choosing $S = V(G) - X$, both $G[S]$ and $G[\bar{S}]$ are connected and satisfy (5.1). It follows that

$$\delta_G S = |S^+| + |S^-| \geq p + q.$$

□

From this theorem we easily obtain some interesting corollaries.

Corollary 5.1.2. *Let G be a graph such that $\Phi_{\mathbb{R}}(G) = p/q + 1$ where p and q are two relatively prime positive integers. Then*

$$|E(G)| \geq p + q + |V(G)| - 2$$

Moreover if G is $2k + 1$-regular

$$|V(G)| \geq \frac{2}{2k - 1}(p + q - 2)$$

Proof. From Theorem 5.1.1, we know that there exists a vertex set S such that $\delta_G(S) \geq p + q$. Both graphs induced by S and $V(G) - S$ are connected, therefore they contain at least $|S| - 1$ and $|V(G) - S| - 1$ edges, respectively. Both these components together contain at least $|V(G)| - 2$ edges and the boundary contains at least $p+q$ edges. Therefore the total number of edges in G is at least $p+q+|V(G)|-2$. Moreover for a $2k+1$-regular graph $(2k+1)|V(G)| = 2|E(G)|$. Therefore

$$|V(G)| \geq \frac{2}{2k-1}(p+q-2)$$

□

This result can be used to get a lower bound of the real flow number of a snark with a given number of vertices.

Theorem 5.1.3. *Let G be a snark containing no more than $8m+4$ vertices. Then*

$$\Phi_{\mathbb{R}}(G) \geq 4 + \frac{1}{m}.$$

Proof. Let $\Phi_{\mathbb{R}}(G) = p/q+1$ where p and q are two relatively prime positive integers. Since G is a snark, we have $p/q > 3$. Suppose $\Phi_{\mathbb{R}}(G) < 4 + \frac{1}{m}$ and therefore $p/q < 3 + 1/m$. Then $q \geq m+1$ and $p > 3m+3$. From Corollary 5.1.2 for 3-regular graphs, we get

$$|V(G)| \geq 2(p+q-2) > 2(4m+2) = 8m+4,$$

which contradicts the fact that G contains no more than $8m+4$ vertices. □

Corollary 5.1.4. *Let G be a snark with $2k$ vertices. Then*

$$\Phi_{\mathbb{R}}(G) \geq 4 + \frac{1}{\lceil \frac{k-4}{4} \rceil}.$$

5.2 Real flow number of Isaacs snarks

The Isaacs snark [8] I_{2k+1} is the graph with the vertex set $V(I_{2k+1}) = \{a_i, b_i, c_i, d_i \mid i = 0 \dots 2k\}$ and the edges

$$E(I_{2k+1}) = \{a_ib_i, a_ic_i, a_id_i, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} \mid i = 0, \dots, 2k\},$$

with indices taken modulo $2k+1$.

In [18] Steffen showed that the Isaacs snark I_{2k+1} has a nowhere-zero $(4+1/k)$-flow, and therefore $\Phi_{\mathbb{R}}(I_{2k+1}) \leq 4 + 1/k$. We show that this inequality is in fact equality.

Theorem 5.2.1. *The real flow number of the Isaacs snark I_{2k+1} is $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$.*

Proof. The proof of the upper bound follows [18]. For a graph G, a directed circuit C in G and a real number r, we define rC to be the flow φ on G such that $\varphi(e) = r$ if $e \in C$ and $\varphi(e) = 0$ otherwise. Moreover, the positive orientation of G with respect to φ induces the orientation of C.

First we define some directed circuits in J_{2k+1}. For $1 \leq i \leq k$ let

$$C_i = b_0, a_0, c_0, d_1, \dots, d_l, c_{l+1}, d_{l+2}, \dots, d_{2i-1}, a_{2i-1}, b_{2i-1}, b_{2i}, b_{2i+1}, \dots, b_0,$$

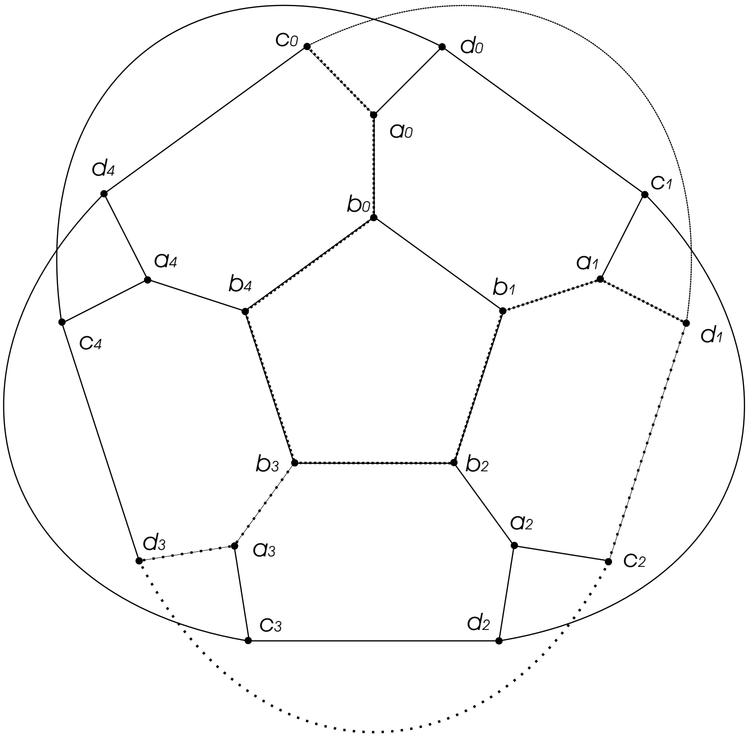

Figure 5.1: The circuits C_1 and C_2 in I_5.

($l < i$, $l \equiv 1 \pmod 2$). For $k+1 \leq i \leq 2k$ let

$$\begin{aligned}
C_i &= b_{2(i-k)-1}, b_{2(i-k)}, a_{2(i-k)}, c_{2(i-k)}, d_{2(i-k)-1}, a_{2(i-k)-1}, b_{2(i-k)-1},\\
C_{2k+1} &= c_0, d_{2k}, a_{2k}, c_{2k}, d_{2k-1}, \ldots, d_0, a_0, c_0,\\
C_{2k+2} &= c_0, d_1, a_1, c_1, d_2, \ldots, d_{2k}, a_{2k}, c_{2k}, d_0, b_0, c_0,\\
C_{2k+3} &= b_0, b_1, \ldots, b_2k, b_0.
\end{aligned}$$

Consider the flows $\varphi_i = (1/k)C_i$ for $i \in \{1, \ldots, k\}$ and the flows $\varphi_i = 1C_i$ for $i \in \{k+1, \ldots, 2k+3\}$. The sum of all these flows is a real nowhere-zero $(4+1/k)$-flow on I_{2k+1}.

Lower bound. Let $G = I_{2k+1}$ where $k \geq 1$ is fixed. Set $r = \Phi_{\mathbb{R}}(G) - 1$. The graph G is a snark and has $8k+4$ vertices. Therefore by Theorem 5.1.3, we have $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$. □

5.3 Remarks

Theorem 5.1.3 shows that a snark G with $8m+4$ vertices cannot have $\Phi_{\mathbb{R}}(G) < 4+1/m$. In the previous section we have showed that there exist graphs with $8m+4$ vertices having the real flow number $4+1/m$. The natural question is how is it with the graphs with $8m-2$, $8m$ and $8m+2$ vertices. Such a graphs with the real flow number $4+1/m$ still can exist. An example of such a graph is Petersen graph. It contains $8.1+2$ vertices and its real flow number is 5. A natural question is whether there are other such graphs. Also we need not require G to be a snark. If it is cubic and has no integral nowhere-zero 4-flow, the statement of Theorem 5.1.3 still holds. Also among these graphs there can be some graphs with $8m-2$, $8m$ or $8m+2$ vertices and real flow number $4+1/m$.

Chapter 6

Summary

A real flow on a graph is a flow on the additive group of reals. The real nowhere-zero r-flow is a real flow φ such that for each dart $1 \leq |\varphi(d)| \leq r-1$. A real flow number $\Phi_{\mathbb{R}}(G)$ of a graph G is the infimum of all reals r such that G has a real nowhere-zero r-flow. The work itself consist of four parts.

In the second chapter we collect some concepts from the graph theory necessary for understanding the work itself. We also introduce the notation used in the work.

In the next chapter we summarize the basics of the real flow theory and we try to unify the terminology of this theory. We prove known results of this theory by using the terms of this theory. Therefore many proofs are original. Perhaps the most interesting is a direct combinatorial proof of the fact, that the infimum from the definition of the real flow number is the minimum.

The fourth chapter contains the first of the original results of this thesis. We continue in the work ok Pan and Zhu [15] who showed that for each real number between 2 a 5 there exists a graph with this real flow number. We answer their question whether for each rational number $4 < q \leq 5$ there exists a snark S such that $\Phi_{\mathbb{R}}(S) = q$. As it is well known snarks are “non-trivial” cubic graphs without 3-edge coloring, and therefore without an integral nowhere-zero k-flow for $k < 5$. To prove this we use the graphs constructed in their work. We present the construction which splits the vertices of degree at least 4 in such way that the newly constructed graph keeps the real flow number, girth and cyclic connectivity. Using this construction we prove that for each rational number $4 < q < 5$ there exists a snark with this real flow number and moreover that there exists infinitely many such snarks. Since Máčajová and Raspaud [13] showed, that there exist infinitely many snarks having real flow number 5, we have proved the following theorem:

Theorem 1. *For every rational number $4 < q \leq 5$ there exist infinitely many snarks with this real flow number.*

In the last part of the thesis we show how small a graph with given real flow number can be. We use the relationship between real flows and orientations of a graph. We get the fact that if a graph G has its real flow number p/q, where p and q are two relatively prime positive integers, G must contain two disjoint connected subgraphs such that the boundary between them contains at least $p+q-2$ edges. If we estimate the number of edges of these two graphs as the number of edges of their spanning trees and use this result for $2k+1$-regular graphs, we get the lower bound of number of vertices of such a graph. In particular, for snarks we can formulate the following theorem.

Theorem 2. *For a snark containing $2m$ vertices*

$$\Phi_{\mathbb{R}}(G) \geq 4 + \frac{1}{\lceil \frac{m-4}{4} \rceil}.$$

Subsequently we show that this estimation can be achieved. We use the result of Steffen [18] who showed that the Isaacs snarks have a real nowhere-zero $(4+1/k)$-flow. Our theorem completes this upper bound. Therefore we get the following theorem:

Theorem 3. *For the Isaacs snark I_{2k+1}, $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$.*

In the estimation in Theorem 2 the equality holds for four factorization classes modulo 8 (the cubic graphs have even number of vertices), namely $8m-2$, $8m$, $8m+2$ a $8m+4$. Isaacs snarks are the largest of these graphs. Only one smaller graph is known – the Petersen graph. These facts raise the following question:

Question 1. *Do there exist snarks, with real flow number $4+1/m$ with either $8m-2$, $8m$, $8m+2$ or $8m+4$ vertices other than the Petersen graph and Isaacs snarks?*

This question could be the objective of further research.

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Súhrn v slovenskom jazyku

Reálny tok na grafe je tok, ktorý používa ako obor hodnôt aditívnu grupu reálnych čísel. Nikde nulový reálny r-tok je taký reálny tok φ, pre ktorý navyše platí $1 \leq |\varphi(e)| \leq r - 1$. Reálne tokové číslo $\Phi_{\mathbb{R}}$ grafu G je infímum z množiny reálnych čísel r, pre ktoré má G nikde nulový reálny r-tok. Práca sa skladá zo štyroch častí.

V druhej kapitole pripomíname čitateľovi niektoré pojmy teórie grafov nevyhnutné pre pochopenie samotnej práce. Zároveň zavádzame niektoré označenia používané v práci.

V ďalšej kapitole zhŕňame základy teórie reálnych tokov a zároveň sa pokúšame zjednotiť používanú terminológiu. Známe vety z teórie tokov dokazujeme pomocou pojmov tejto teórie. Preto sú mnohé dôkazy známych tvrdení nové. Asi najzaujímavejším z nich je nový priamy kombinatorický dôkaz skutočnosti, že infímum z definície reálneho tokového čísla sa dosahuje.

Štvrtá kapitola obsahuje prvý z vlastných výsledkov práce. Nadviažeme na prácu Pana a Zhu, ktorí ukázali, že pre každé racionálne číslo medzi 2 a 5 existuje graf s týmto reálnym tokovým číslom. Kladne zodpovedáme otázku Pana a Zhu [15], či pre každé racionálne číslo $4 < q \leq 5$ exiskuje snark S, pre ktorý platí $\Phi_{\mathbb{R}}(S) = q$. Ako je známe, snarky sú "netriviálne" kubické grafy bez hranového 3-farbenia, a teda bez nikde nenulového celočíselného k-toku pre $k < 5$. Na dôkaz používame nimi vytvorené grafy. Prezentujeme konštrukciu, ktorá rozdeľuje vrcholy grafu stupňa aspoň 4, tak že novovzniknutý graf si zachováva tokové číslo, obvod i cyklickú súvislosť. Tým dokážeme, že pre každé racionálne číslo $4 < q < 5$ exiskuje snark s týmto reálnym tokovým číslom a dokonca, že existuje takýchto snarkov nekonečne veľa. Keďže Máčajová a Raspaud [13] ukázali, že existuje nekonečne veľa snarkov s reálnym tokovým číslom 5, dokazali sme nasledujúcu vetu:

Veta 1. *Pre každé racionálne číslo $4 < q \leq 5$ exiskuje nekonečne veľa snarkov s týmto reálnym tokovým číslom.*

V poslednej časti práce ukazujeme, akú najmenšiu veľkosť môžu mať grafy s daným tokovým číslom. Použitím vzťahu medzi orientáciami grafu a reálnym tokovým číslom dostávame, že ak má graf reálne tokové číslo p/q, kde p a q sú nesúdeliteľné kladné celé čísla, tak v ňom musia existovať dva disjunktné súvislé podgrafy s hranicou aspoň $p + q - 2$. Ak počet hrán týchto podgrafov odhadneme ich kostrou a výsledok aplikujeme na $2k + 1$ regulárne grafy získame dolný odhad pre počet vrcholov grafu. Špeciálne pre snarky môžeme formulovať nasledovnú vetu:

Veta 2. *Pre snarky s $2m$ vrcholmi platí*

$$\Phi_{\mathbb{R}}(G) \geq 4 + \frac{1}{\lceil \frac{m-4}{4} \rceil}.$$

Následne ukazujeme, že tento odhad sa dosahuje. Využijeme výsledok Steffena [18], ktorý ukázal, že Isaacsove snarky majú nikde nulový reálny $(4 + 1/k)$-tok. Naša veta dopĺňa tento odhad, čím dostávame nasledujúce tvrdenie:

Veta 3. *Pre Isaacsov snark I_{2k+1} platí,* $\Phi_{\mathbb{R}}(I_{2k+1}) = 4 + 1/k$.

V odhade vo Vete 2 však nastáva rovnosť pre 4 zvyškové triedy modulo 8 (kubické grafy majú párny počet vrcholov) a to $8m - 2$, $8m$, $8m + 2$ a $8m + 4$. Isaacsove snarky sú teda najväčšie z týchto grafov. Z menších grafov je známy iba Petersenov graf. Tieto fakty vedú k nasledujucej otázke:

Otázka 1. *Existujú aj ďalšie snarky s reálnym tokovým číslom* $4 + 1/m$ *a* $8m - 2$, $8m$, $8m + 2$ *alebo* $8m + 4$ *vrholmi, iné ako Petersenov graf a Isaacsove snarky?*

Táto otázka by mohla byť predmetom ďalšieho výskumu.