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Broadcasting on Butterfly Network with Dynamic Faults

(Diploma thesis)

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I do declare that I elaborated this submitted thesis on my own using only literature listed.

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Abstract

Unlike *localized* communication failures that occur on a fixed (although a priori unknown) set of links, *dynamic* faults can occur on any link. Known also as mobile or ubiquitous faults, their presence makes many tasks difficult if not impossible to solve even in synchronous systems. Analysis of broadcasting in model with dynamic faults is considered in shouting communication mode in which any node of network can inform all its neighbors in one time step. During each time step number of faulty links can be less then edge-connectivity of network. The problem is to find an upper and lower bound on number of time steps necessary to complete broadcasting.

We prove that for k-ary butterfly network of dimension n lower bound is $3n$ and upper bound $3n + O(\log_k n)$.

We also prove that k-ary wrapped butterfly network of dimension n has lower bound $\frac{3}{2}n + 1$ and upper bound $\frac{3}{2}n + O(\log_k n)$.

Contents

1 Introduction

Broadcasting is problem of dissemination of message in entire network of computers or processors. Broadcasting is fulfilled, if all nodes in network are informed. Broadcasting might be one-message only or with simultaneous messages, where sending one message forbids sending other message. Problem of broadcasting multiple messages in network is standalone problem, even without any fault.

There are many views on how broadcasting is accomplished. The message can be sent simultaneously too many neighbors or just to one neighbor at a time. Broadcasting message to one neighbor at a time is another standalone problem, though some results for broadcasting with faults exists.

In this work all communication is synchronous. In each time step any node of network can broadcast message to all its neighbors. This is also known as shouting model or gossiping. Without any error in communication, message is broadcasted to every node in network in time equal to diameter $diam(G)$ of network G. Thus we make this value our referential value when considering faults.

As fault we denote impossibility to transfer message between two adjoint nodes. If faults are static, message can be broadcasted also in case of many faults. So many that just Hamiltonian path is non-faulty. But when faults can change in time, we have to be more restrictive on number of faults that occur in one time step. And we state condition there must not be more faults than edge-connectivity of graph. This condition is also noted as fault-tolerance.

One of longterm aim in research of broadcasting is to find topology that will be maximaly fault tolerant, with minimal broadcast time and lowest number of edges for defined number of vertices. First comprehensive research in boradcasting with dynamic faults ws done on hypercubes by De Marco and Vaccaro, who proved that message is broadcasted in worst case in more than $diam(n)+2$ an less than $diam(n) + 7$ time steps.[DV98]. For this they mainly used hypercube's recursive character, where each n-dimensional hypercube consists of non-interleaving k-dimensional hypercubes. Later on Dobrev and Vrt'o refined this result for hypercubes proved that the worst case is exactly $diam(n)+2$ time steps, using very tight estimation of isometric number.[DV99]

Hypercubes are graphs not considered optimal, mostly when we look at graph degree or diameter. As seen in tables Table 1. and Table 2. some bound degree graphs have better characteristics. In later work Dobrev and Vrt'o proved that even on Even Tori, which is bound degree graph, the limit for broadcasting message is $diam(n) + 2$.[DV00]

The question this work tries to answer is: "How are Butterfly networks fault-tolerant?" and "What time is needed to broadcast message in butterfly network?"

deg	Wrapped Butterfly	Hyper-cube	de Bruijn
2	31	-	20
3	20	-	13
4	16	-	10
10	10	-	6
20	8	20	5
50	7	-	4
100	5	-	3

Table 1: Graph diameter for $|V| = 10^6$. (Cells with a dash indicate that the graph does not support the corresponding degree)

deg	Wrapped Butterfly	Hyper-cube	de Bruijn
2	22.4	-	17.9
3	14.7	-	11.7
4	11.8	-	9.4
10	7.3	-	5.8
20	5.7	10	4.5
50	4.3	-	3.5
100	3.65	-	2.98

Table 2: The average distance in each graph for $|V| = 10^6$.

[LKRG03]

2 Butterfly network

Butterfly network is bound degree network topology. This topology was used in ATM switches. Butterfly network is also known as Banyan network, or with some modification it become Benes network.

Definition 2.1 (Butterfly network)

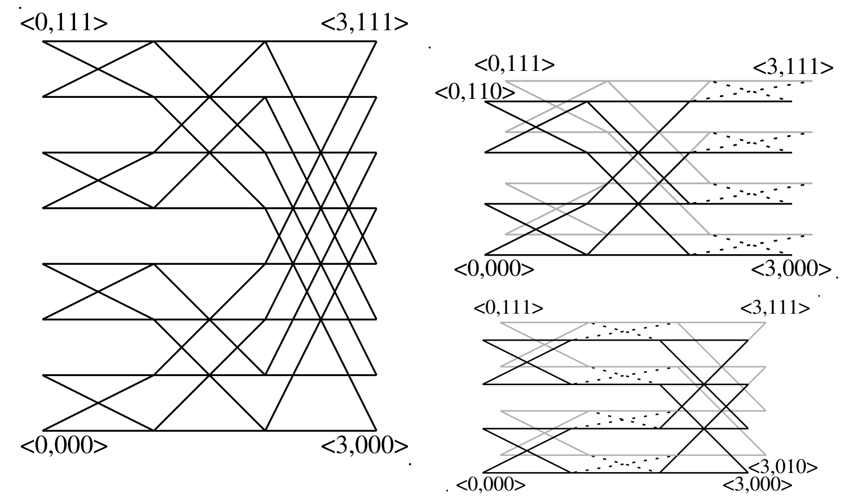
The *k-ary Butterfly network BF(k,n) of dimension n* is a graph of vertex-set $V = \{0, 1, \ldots, n\} \times \{0, 1, \ldots, k-1\}^n$, where $\{0, 1, \ldots, k-1\}^n$ denotes the set of "length n k-ary strings". For each vertex $v = \langle i, \alpha \rangle \in V$, $i \in \{0, 1, \ldots, n\}$, $\alpha \in \{0, 1, \ldots, k-1\}^n$, we call i the level and $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ position within level string of v. Each vertex $v = \langle i, \alpha \rangle \in V$ for $i < n$ and any α is connected with vertex $v' = \langle i', \alpha' \rangle$, where $i' = i + 1$, $\alpha'_i \in \{0, 1, \ldots, k-1\}$, $\alpha_j = \alpha'_j \; \forall j \neq i$

There are some basic properties of Butterfly network we keep using. First of all complete k-ary tree can be embedded into k-ary Butterfly network [KMPS92].

Definition 2.2

By $T_n\langle 0, \mathbf{0} \rangle$ we understand complete k-ary tree with root vertex $\langle 0, \mathbf{0} \rangle$ which has k^i vertices at each level i.

Next important properties of butterfly networks is their recursive nature. If you chose any number of levels we can find sub-butterfly of dimension equal to chosen number. There is not just one particular sub-butterfly, but there are sub-butterflies that partition these levels into vertex disjoint sub-butterflies.



Picture 1: Leftmost picture shows standard picture of BF(2,3). On the right side up there is BF(2,3) drawn such that levels $0, 1, 2$ form BF(2,2) and levels $2, 3$ form BF(2,1). Picture on right down shows partitioning of BF(2,3) to BF(2,1)'s.

If we partition butterfly into sub-butterflies with levels less than i, and also sub-butterflies with all levels more or equal to i, we will found out that there is only one vertex any of these "concurrent" sub-butterflies can connect. We will also come to knowledge that this one vertex is present for any "concurrent" sub-butterflies. This is very powerful knowledge and like in proof for hypercubes [DV98] we will use this knowledge extensively.

Definition 2.3 (Sub-butterfly)

Let $\langle i, \alpha \rangle$ be vertex of BF(k,n). Let us denote by BF$_{\langle i,\alpha \rangle}$(k,m), where $i+m \leq n$ subgraph of graph BF(k,n) with vertices $\langle j, \beta \rangle \in BF(k, n)$ where $i - 1 < j < i + m$ and $\beta = \alpha_0 \dots \alpha_{i-1}\{0, 1, \dots, k-1\}^j \alpha_{i+j+1} \dots \alpha_n$

BF$_{\langle i,\alpha \rangle}$(k,m) is isomorphic to BF(k,j). Isomorphism $h(\langle l, \beta \rangle) = \beta_i, \dots, \beta_{i+j}$

(Need to rewrite) Any BF$_{\langle 0,\alpha \rangle}(k, m) \subseteq$ BF(k,n) partition() level j ($j \leq n$) of BF(k,n) into 2^m distinct sub-butterflies of type BF$_{\langle m,\alpha' \rangle}$(k,n-m).

$\ll$ Tu je potrebne vlozit obrazok ako moze vyzerat BF, kde je vidno ten bod Picture 1. $\gg$

2.1 Lower Bound

In this section we re going to show that only one dynamic fault is sufficient to slow broadcasting of message from vertex $\langle 0, \mathbf{0}\rangle$ by 50%. First of all we construct path which is marked as "faulty". This "faultiness" can be assured by leaving each edge faulty, since one of its vertex is informed, until both of its vertices are informed. We simultaneously prove, there is only one edge in "faulty" path that has exactly one informed vertex.

Definition 2.4 *By $P_{\mathbf{0}}$ we understand a path in BF(k,n) such that*

$$P_{\mathbf{0}} = \langle 0, \mathbf{0}\rangle\langle 1, \mathbf{0}\rangle \dots \langle n, \mathbf{0}\rangle$$

Lemma 2.1 *For any BF(k,n) there exists path of length 3n edge-disjoint to $P_{\mathbf{0}}$ completely inside BF(k,n).*

Construction
Path $\langle 0, \mathbf{0}\rangle, \dots, \langle n+1, \mathbf{1}\rangle, \dots, \langle 0, \mathbf{1}\rangle, \dots, \langle 0, \mathbf{0}\rangle$ is such path.

The next lemma proves that if part of the constructed "faulty" path is completely inside a certain sub-butterfly, the broadcasting of the message to the vertices of this part of path can be influenced only by the considered sub-butterfly. That is, there is no "miraculous" shortcut using the rest of the butterfly that would speed-up the broadcasting.

Lemma 2.2 *Shortest path from $\langle first, \mathbf{0}\rangle$ to $\langle last, \mathbf{0}\rangle$ edge-disjoint to $P_{\mathbf{0}}$ is wholly inside $BF_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})$.*

Proof Let be path $P = P_0, \dots, P_{min}$ the shortest path from $\langle first, \mathbf{0}\rangle$ to $\langle last, \mathbf{0}\rangle$ edge-disjoint to $P_{\mathbf{0}}$. By contradiction let us assume, there exists vertex within P which is outside of $\mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})$. Let be v_i such that $v_i \in \mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})$ and $v_{i+1} \notin \mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})$. Let be v_j such that $j = min_k\{k > i, v_k \in \mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0}), v_{k-1} \notin \mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})\}$. Subpath $v_{i+1}, \dots, v_{j-1}$ is wholly outside $\mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})\}$. By Induction on number of vertices $K = |v_k|$ (where $i < k < j$ and $first \leq level(v_k) \leq last$) we prove that there exists shorter path from v_i to v_j.

1. $K = 0$. There **is no** vertex v_k such that $first \leq level(v_k) \leq last$.

Let us assume that $level(v_i) = last$ (for $level(v_i) = first$ proof is similar) thus $\forall k, i \leq k \leq j$, $level(v_k) > last$. We know that there is no edge $\langle level - 1, \alpha\rangle, \langle level, \alpha'\rangle \in v_i, \dots, v_j$, for $level \leq last$. Thus v_i and v_j are equal on first $last$ places. From $v_i, v_j \in \mathrm{BF}_{\langle first,\mathbf{0}\rangle}(last - first, \mathbf{0})$ we know that v_i, v_j are equal on places $last, \dots, n$. So $v_i = v_j$ is the shorter path.

2. $K > 0$. For any $K' < K$ we know that there is shorter path for v_i to v_j if there is K' vertices v_k where $first < level(v_k) < last$.

For this case we will use different definition of path $P = e_0, ..., e_n$. In this definition path $\vec{E}(v_0)$ is described by starting position v_0 and sequence of directions $\vec{E} = \vec{e_1}, ..., \vec{e_n}$. Direction $\vec{e_i} = (\vec{l_i}, \vec{\alpha_i})$ is normalized edge $e_i = (v_{i-1}, v_i) = (\langle\alpha_{i-1}, l_{i-1}\rangle, \langle\alpha_i, l_i\rangle)$ in a way that it describes change in level $\vec{l_i} = l_i - l_{i-1}$ and change in position within level $\vec{\alpha_i}$ where $(\alpha_i)_L \oplus_k \vec{\alpha_i} = (\alpha_{i-1})_L$ where $L = \min\{l_{i-1}, l_i\}$. [1]

For any vertex $\langle\beta, l\rangle$ we can obtain next vertex $\langle\beta', l'\rangle$ by applying $\vec{e_i}$ from which $\vec{e_i}(\langle\beta, l\rangle) = \langle\beta', l'\rangle$, where $l' = l + \vec{l_i}$, $(\beta)_L = (\beta)'_L \oplus_k \vec{\alpha_i}$ and $\forall j \neq L, (\beta)_j = (\beta)'_j$, $L = \min\{l_{i-1}, l_i\}$.

This definition allow us to easily exchange starting vertex of path without changing directions of path. This change can be done in case that directions don't lead us outside of BF(k, n). Changing position within level never leads to leaving BF(k, n). Changing level could, in case any i within path $\sum_i \vec{l_i} + level$ ($level$ of new starting vertex) is bigger than n or less than 0.Using alternative starting vertex with level equal to the original vertex is enough to assure that the new path is inside BF(k, n).

Now we apply the new definition on subpath $v_i, \ldots, v_j$. Let be v_k vertex such that $level(v_{k-1}) - 1 = level(v_k) = level(v_{k+1}) + 1$ in subpath $v_i, \ldots, v_j$. Let be $\vec{E_1}(v_i), \vec{E_2}(v_k)$ path $v_i, \ldots, v_j$ in our new definition.

We change order of the directions of two identified subpaths. $\vec{E_2}(v_i), \vec{E_1}(v'_k) = v'_i, \ldots, v'_n$ we know that the vertex $v'_{i+1} \in$BF$_{\langle first, \mathbf{0}\rangle}(last - first, \mathbf{0})\}$. And if we prove that $v'_n = v_n$ we constructed path with less than K vertices outside BF$_{\langle first, \mathbf{0}\rangle}(last - first, \mathbf{0})\}$ within $v_i, \ldots, v_j$ and fulfilled induction assumption.

Level of v_j is equal to level of v_i plus sum of all changes in level within subpath $v_i, \ldots, v_j$ $level(v_j) = level(v_i) + \sum_i \vec{l_{1i}} + \sum_i \vec{l_{2i}}$.

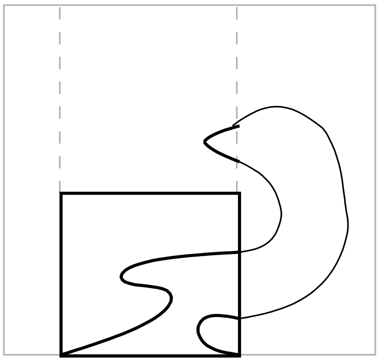
Level of v'_j is equal to level of $v'_i = v_i$ plus sum of all changes in level within subpath $v'_i, \ldots, v'_j$. $level(v'_j) = level(v_i) + \sum_i \vec{l_{2i}} + \sum_i \vec{l_{1i}}$.

From commutativity of "+" we know that $level(v_n) = level(v'_n)$ Proving that position within level is equal is shown equally, by summing all the partial changes (directions) in path. vertices v_i, v_k and v_n are in the same level. v'_l is also in the same level. Thus we sum the same direction in the same level for $v_i, \ldots, v_n$ and $v'_i, \ldots, v'_n$.

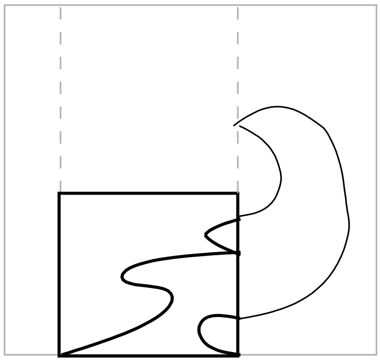
$\bowtie$

By next lemma we prove that transmitting information along path that is edge-disjoint to "faulty" one, needs to be least $3n$ time steps long. This proves that it is not much better to broadcast message far away from "faulty" path than to wait for broadcasting message from one vertex of "faulty" path to another by circles of length 4 that are present in BF$(k, 1)$.

[1] $x \oplus_k y = x + y$ (modulo k)



Picture 2: Potentially shortest path that from $\langle first, \mathbf{0}\rangle$ to $\langle last, \mathbf{0}\rangle$



Picture 3: Path with less vertices outside sub-butterfly.

In last lemma of this section all these informations are put together to prove that message need not to be broadcasted from $\langle 0, \mathbf{0}\rangle$ to $\langle n, \mathbf{0}\rangle$ in less than $3n$ time steps.

Lemma 2.3 *Shortest path $P = \langle first, \mathbf{0}\rangle, \ldots, \langle last, \mathbf{0}\rangle$ edge disjoint to P_0 has length $3(last - first)$.*

Proof From Lemma 2.2 we know that no part of path P is in levels bigger than $last$ and lower than $first$. So we count number of edges of P between levels i and $i+1$ for $first \leq i < last$. We show that there are at least three edges from level i to level $i+1$ in P thus length of path P is at least $3(last - first)$.

In all levels from $first$ to $last - 1$ there is at least one edge in P to higher level. This is mandatory for path P to get from level $first$ to level $last$. The only edge from level i to level $i+1$ cannot be path not changing **bit**. If so it had to be from $\langle i, \mathbf{0}\rangle$ to $\langle i+1, \mathbf{0}\rangle$ to get from $\langle first, \mathbf{0}\rangle$ to $\langle last, \mathbf{0}\rangle$ which is part of P_0 thus violating definition of P. In case the edge from level i to level $i+1$ is changing ***i*-th bit** there is another edge to redo this change. If there are just two edges for level i it means that path begins and ends in a level less than level i thus there is at least third edge.

We know that for $\forall i, first \leq i < last$ there are at least 3 edges. This shows us the length of P is at least $3(last - first)$. From Lemar 2.1 we know there exists lemma of length $3(last - first)$.

$\bowtie$

Lemma 2.4 *Broadcasting on BF(k,n), with one dynamic fault, cannot be always completed in less than $3n$ time steps.*

Proof Let be $\langle 0, \mathbf{0}\rangle$ initiator of broadcasting in time step 0. Let be edge $\langle \lfloor \frac{i}{3} \rfloor, \mathbf{0}\rangle$,$\langle \lfloor \frac{i}{3} \rfloor + 1, \mathbf{0}\rangle$ faulty in time step i. We prove that vertex $\langle n, \mathbf{0}\rangle$ won't be informed in time step less than $3n$.

We prove by induction that in time step, less then $3i$, vertex $\langle i, \mathbf{0} \rangle$ won't be informed.
$i = 1$: BF(k,1) is equal to $K_{k,k}$. In $K_{k,k}$ all cycles have length 4. In time steps 1 and 2 edge $\langle 0, \mathbf{0} \rangle, \langle 1, \mathbf{0} \rangle$ is faulty the only way to inform $\langle 1, \mathbf{0} \rangle$ is by going around by some of length 4. Thus leaving $\langle 1, \mathbf{0} \rangle$ uninformed for time steps 1, 2.
$i \to i+1$: For any $0 < j \leq i$ informing vertex $\langle j, \mathbf{0} \rangle$ cannot be done in less than $3j$ time steps (IA). Informing vertex $\langle i+1, \mathbf{0} \rangle$ from $\langle j, \mathbf{0} \rangle$ cannot be done in less than $3(i-j+1)$ if the faulty edge is $\langle \left\lfloor \frac{j'}{3} \right\rfloor + j, \mathbf{0} \rangle, \langle \left\lfloor \frac{j'}{3} \right\rfloor + j + 1, \mathbf{0} \rangle$ Faulty edge in time steps $3j + j'$ is $\langle \left\lfloor \frac{j'+3j}{3} \right\rfloor + j, \mathbf{0} \rangle, \langle \left\lfloor \frac{j'+3j}{3} \right\rfloor + 1, \mathbf{0} \rangle$ which is the same as the needed one. So vertex $\langle i+1, \mathbf{0} \rangle$ cannot be informed in less than $3j + 3(i-j+1) = 3(i+1)$ time steps through path P passing through vertex $\langle j, \mathbf{0} \rangle$.
So we have only one another possibility to be faster than $3(i+1)$ its by path P vertex disjoint to path P_0. But length of this path is by Lemma 2.3 at least $3(i+1)$. Leaving no other choice.

$\bowtie$

2.2 Upper bound

To begin we prove some basic lemmas. First lemma tell us that if there is informed level within butterfly there are no obstacles in broadcasting message throughout whole butterfly.
Second one proves that broadcasting within smallest butterfly is done in less than 4 time steps.

Lemma 2.5 *If there is i-th level of BF(k,n) completely informed in time step T. Whole BF(k,n) is informed in time step $T + \max\{n-i, i-1\}$*

Proof By induction we prove that in time step $T + j$, levels $\{i-j, \ldots, i+j\}$ are informed. For each vertex $v = \langle i+j, \alpha \rangle$ ($\langle i-j, \alpha \rangle$) there exists k distinct neighbor vertices in level $i-(j-1)$ ($i+(j-1)$), by induction assumption informed in time step $j-1$. There are at most $k-1$ faults. So there it is at least one correct channel for each v to informed vertex in time step $T+j$.
From this statement lemma holds.

$\bowtie$

Lemma 2.6 *The number of steps needed for broadcasting in BF(k,1) is* 4.

Proof BF(k,1) is equal to $K_{k,k}$ thus we can assume without loss of generality that initiator is $\langle 0, \mathbf{0} \rangle$.

In time step 1 there it is at least another informed vertex within level 1. Now there are $2(k-1)$ distinct vertices neighbor to informed ones.

In time step 2 we have $k+1 = 2 + (2(k-1) - (k-1))$ informed vertices. Further we can assume that, no level is fully informed. Cause the opposite means full knowledge in time step 3 by Lemma 2.5.

In time step 3 there is one fully informed level. By contradiction. Let us assume that there are non informed vertices $v_0 = \langle 0, \alpha \rangle$ and $v_1 = \langle 1, \beta \rangle$. This means that number of vertices informed in time step 2 neighbor to v_0 or v_1 is less or equal to $k-1$ (number of possible faults). Together v_o and v_1 have connection to any vertex in BF(k,n). So there are $k+1 > k-1$ informed vertices in time step 2 neighbor to v_0 and v_1. which contradicts to our assumption.

By Lemma 2.5 we know that BF(k,1) is wholly informed in 4 steps.

$\bowtie$

These two lemmas are sufficient to prove upper bound for butterfly network. Next lemma shows us what is the needed time if we use only these two knowledges.

Lemma 2.7 *Broadcasting in BF(k,n) is completed in $5n-1$ time steps.*

Proof Our first step is to prove this Lemma in special case only, where initiator of broadcasting lies within level 0 or n. These two cases are similar for proving, we just need to change each level for its opposite level $i \to n-i$.
We can assume that initiator is $\langle 0, \mathbf{0} \rangle$. We know that in time step 4 there is informed $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,1)$ (Lemma 2.6). Let us assume, that all vertices in levels $i-1$ and i ($i < n$) are informed in $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i)$ in time step $4(i+1)$. Each vertices in level $i+1$ in $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i+1)$ are adjacent to a vertex which lies within level i in $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i)$. And vertices in level i are adjacent to vertices in level $i+1$ within $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i+1)$. Adjoining vertices within level i and $i+1$ lies in the same $\mathrm{BF}_{\langle \alpha,i\rangle}(k,1)$ where $\langle \alpha, i \rangle \in \mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i)$. By reusing Lemma 2.6 we know that all vertices in these sub-butterflies $\mathrm{BF}_{\langle \alpha,i\rangle}(k,1)$ are informed in time step $4(i+1)+4$. Thus all vertices in level i and $i+1$ within $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}(k,i+1)$ are informed in time step $4((i+1)+1)$.
Now we know that in time step $4n$ there are all vertices inside levels n and $n-1$ informed in $\mathrm{BF}(k,n)$. By Lemma 2.5 we obtain that in $n-1$ more time steps there are all vertices informed in $\mathrm{BF}(k,n)$. Thus $\mathrm{BF}(k,n)$ is informed in $4n+n-1 = 5n-1$ time steps. To complete this Lemma we need to prove that for any initiator $\langle i, \alpha \rangle$ butterfly $\mathrm{BF}(k,n)$ is informed in $5n-1$ time steps.
From Butterfly properties we know that there are butterflies $\mathrm{BF}_{\langle 0,\alpha\rangle}(k,i)$ and $\mathrm{BF}_{\langle i,\alpha\rangle}(k,n-i)$ for which initiator $\langle i, \alpha \rangle$ is in first (or last) level. For these Butterflies we already know that they are informed in $5i-1$ respectively $5(n-$

$i) - 1$ time steps.
Last vertices in $\mathrm{BF}_{\langle 0,\alpha\rangle}(k,i)$ are first vertices for all $\mathrm{BF}_{\langle i,\alpha'\rangle}(k,n-i)$. First vertices in $\mathrm{BF}_{\langle i,\alpha\rangle}(k,n-i)$ are last vertices in $\mathrm{BF}_{\langle 0,\alpha'\rangle}(k,i)$. Information in $\mathrm{BF}_{\langle i,\alpha'\rangle}(k,n-i)$ is broadcasted in $5(n-i)-1$ time steps from broadcasting in $\mathrm{BF}_{\langle 0,\alpha\rangle}(k,i)$ which is in $5i-1$ time steps. Thus all butterflies $\mathrm{BF}_{\langle i,\alpha'\rangle}(k,n-i)$ are informed in $5(n-i)+5i-1-1 < 5n-1$ time steps. For butterflies $\mathrm{BF}_{\langle 0,\alpha'\rangle}(k,i)$ this is proved equally.

$\bowtie$

This Lemma is solution rather non-optimal. It proves that information in faulty Butterfly is broadcasted in less than $2.5diam(n)-1$ time stamps. So we had to find better solution. The solution we follow uses embedded k-ary tree. In next lemma we find out how the message is disseminated within k-ary tree with $k-1$ dynamic faults.

Lemma 2.8 *Let be $\langle 0, \mathbf{0}\rangle$ initiator of broadcasting. In time step $t \leq n$ there is at least*

$$S(t) = 1 + \sum_{j=0}^{t-1} k^j$$

informed vertices in $T_n\langle 0, \mathbf{0}\rangle$

Proof This we prove by counting number of vertices that can be informed in time step $j+1$ but not informed in time step j. Let us denote this by $S'(j)$ Let $S'(j) = k^j + k - 1$.
$S'(0) = k^0 + k - 1 = k$, which is degree of $\langle 0, \mathbf{0}\rangle$. Number of vertices newly informed in time step j is $S'(j-1) - (k-1)$ this is number of vertices that could be informed in time step j without maximal number of possible faults. For each vertex newly informed in time step j there are k new vertices that could be informed in time step $j+1$. So $S'(j+1) = k(S'(j) - k + 1)$ and the $k-1$ vertices that were cut of by faults.

Without loss of generality we can assume exactly $k-1$ active faults. If there is less then $k-1$ channels down, there would be k new vertices instead of each correct channel.

So $S'(j+1) = k(S'(j)-k+1)+k-1 = k(k^{j-1}+k-1-k+1)+k-1 = k^j+k-1$.

$S(t)$ can be counted as sum of $S'(0)-(k-1), \ldots, S'(t-1)-(k-1)$ plus initiator.

$\bowtie$

Now we know how many vertices are informed within embedded tree. Though this knowledge doesn't help us much. Thus we prove also how many vertices are informed in last but one level within this tree.

Lemma 2.9 *Let be $\langle 0, \mathbf{0}\rangle$ initiator of broadcasting. In time step $t \leq n$ there is at least $\frac{k^{t-1}}{k+1}$ informed vertices at level $t-1$*

Proof This we prove by contradiction. Let us consider there exists $k^{t-1} - \frac{k^{t-1}}{k+1} + 1$ uninformed vertices in $T_t\langle 0, \mathbf{0}\rangle$. For each uninformed vertex $v = \langle t-1, \alpha\rangle$ there are k vertices, not informed within level t cause the only way how to reach these vertices in time t is through uninformed vertex v. So we have in $T_t\langle 0, \mathbf{0}\rangle$

$$(k+1)(k^{t-1} - \frac{k^{t-1}}{k+1} + 1) = k^t + k^{t-1} - k^{t-1} + k + 1 = k^t + k + 1$$

uninformed vertices. From Lemma 2.8 we have there exists $S(t)$ informed vertices in $T_t\langle 0, \mathbf{0}\rangle$. There should be $S(t) + k^t + k + 1 = \sum_{j=0}^{t} k^j + k + 2$ vertices in k-ary tree $T_t\langle 0, \mathbf{0}\rangle$ with depth t. But full k-ary tree of depth t has $\sum_{j=0}^{t} k^j$ vertices.

$\bowtie$

After proving that we know of $\frac{k^{t-1}}{k+1}$ informed vertices in one level we can have a look at sub-butterflies BF(k, s) that are attached to these vertices. If there is more sub-butterflies BF(k, s) than number of faults that can occur, in next s time steps, we know that at least one sub-butterfly will be informed.
After next s steps whole BF(k, s) is informed. From this informed sub-butterfly we can easily inform complete butterfly.
Next lemma shows what is the s and details of broadcasting message to fully informed BF(k, n).

Lemma 2.10 *Any vertex on level 0 can broadcast message within BF(k,n) in time $3n + 3\log n + 5$*

Proof Without loss of generality we can assume that initiator is $\langle 0, \mathbf{0}\rangle$.
By Lemma 2.9 we know that in time step $\lceil \log n\rceil + 3 \leq n$ there is $\frac{k^{\lceil \log n\rceil + 2}}{k+1} > \frac{nk^2}{k+1}$ informed vertices. In Time steps $\lceil \log n\rceil + 4, \ldots, n+1$ there is possible to encourage $(n - \lceil \log n\rceil - 3)(k-1)$ faults. Number of informed vertices is greater then number of possible faults.

$$\begin{aligned} \frac{nk^2}{k+1} &> (n - \lceil \log n\rceil - 3)(k-1) \\ nk^2 &> (n - \lceil \log n\rceil - 3)(k^2-1) \end{aligned}$$

Cause

$$\begin{aligned} n &> n - \lceil \log n\rceil - 3 \\ k^2 &> k^2 - 1 \end{aligned}$$

Thus whole level $n - \lceil \log n \rceil + 2$ in $BF_{\langle \lceil \log n \rceil + 2, \alpha \rangle}(k, n - \lceil \log n \rceil + 2)$ for some α is informed in time step $n + 1$
By Lemma 2.5 we have in time step $n + 1 + n - (\lceil \log n \rceil + 2) = 2n - \lceil \log n \rceil - 1$ fully informed $BF_{\langle \lceil \log n \rceil + 2 \rangle}(k, n - \lceil \log n \rceil + 2, \alpha)$
So in time step $2n - \lceil \log n \rceil - 1$ there is at least one informed vertex for $\mathrm{BF}_{\langle \lceil logn+2 \rceil, \alpha \rangle}(k, \lceil \log n + 2 \rceil)$ for any α.
In time step $2n - \lceil \log n \rceil - 1 + 4(\lceil \log n \rceil + 1)$ there is wholly informed $\mathrm{BF}_{\langle 0, \beta \rangle}(k, 1)$ for any β. By Lemma 2.6 there is at most 4 time steps needed for broadcasting the message in whole BF(1,k). This means k informed vertices in 1st and 2nd level and in next 4 time steps k^2 informed vertices in 2nd and 3rd level. ...
Thus in time step $2n + 3 \lceil \log n \rceil + 3$ we have fully informed level 0 and 1. And by lemma 2.5 in time step $2n + 3 \lceil \log n \rceil + 3 + (n - 1) = 3n + 3 \lceil \log n \rceil + 2$ whole BF(k,n) is informed.

$\bowtie$

To finalize upper bound we need to know number of time steps needed to broadcast message from any vertex. From any vertex v we see two sub-butterflies one containing level 0 the other level n which have v on their marginal level. Broadcasting in these sub-butterflies can be completed by previous lemma.
After informing these two sub-butterflies we know that all other sub-butterflies within same levels as these two are sharing vertex with informed sub-butterflies. Thus using previous lemma once more is sufficient to broadcast message to all vertices within $\mathrm{BF}(k, n)$
Next lemma count number of time steps needed to broadcast message from any vertex to all vertices performed by this procedure.

Lemma 2.11 *Broadcasting on BF(k,n) is completed in* $3n + 6 \log(\frac{n}{2}) + 4$

Proof Let be $\langle i, \alpha \rangle$ initiator of broadcasting. $\langle i, \alpha \rangle$ is marginal vertex for sub-butterflies $\mathrm{BF}_{\langle 0, \alpha \rangle}$(k,i) and $\mathrm{BF}_{\langle i, \alpha \rangle}$(k,n-i).

By Lemma 2.10 information is broadcasted in $\mathrm{BF}_{\langle i, \alpha \rangle}$(k,n-i) in $3(n - i) + 2 \log (n - i) + 2$ time steps.
From Butterfly properties we have that for any $\mathrm{BF}_{\langle 0, \beta \rangle}$(k,i) there is one informed vertex. By Lemma 2.10 information is broadcasted in $\mathrm{BF}_{\langle 0, \beta \rangle}$(k,i) in $3i + 3 \log i + 2$ time steps. In time step $3(n - i) + 3 \log (n - i) + 2 + 3i + 3 \log i + 2 = 3n + 3(\log (n - i) + \log i) + 4 \leq 3n + 6 \log \frac{n}{2} + 4$.

For $\mathrm{BF}_{\langle 0, \alpha \rangle}(k, i)$ it is the same situation as above. Using $\mathrm{BF}_{\langle i, \beta' \rangle}$(k,n-i).

$\bowtie$

3 Wrapped Butterfly Network

Wrapped Butterfly network is bound degree network topology. The k-ary wrapped butterffly network of dimension n can be obtained from k-ary Butterfly Network of dimension $n+1$ by merging vertices $\langle 0, \alpha \rangle$ and $\langle n, \alpha \rangle$. Thus wrapped butterfly network maintains the same properties as Butterfly Networks do.

Complete k-ary Tree can be embedded into Wrapped Butterfly Network. Unlike in Butterfly network where embedding Tree of depth n is possible only with root in level 0 and n, complete k-ary Tree of depth n can be embedded in wrapped butterfly network with root in any vertex.
There is also possibility to embed cube connecet cycle network into wrapped butterfly[FU92]

Merged vertices $\langle 0, \alpha \rangle$ have $2k$ neighbors and as such all vertices have degree $2k$. Butterfly network has Cayley Graph topology. Binary Wrapped Butterfly has been resolved as 'New Family Cayley graph' [CL97].

There was research in broadcasting on Butterfly topology where each vertex can be transmit a message to exactly one vertex to which it is adjacent during one time step, and each vertex can either transmit or receive the message per time step. where

Though wrapped butterfly network is Cayley graph, for some proof we need to distinguish opposite directions. One direction we denote as up and the opposite down. New terms uplink, up-tree, up sub-butterfly and downlink, down-tree, down sub-butterfly we can use to easily distinguish direction. Up-tree T_s^+ is complete k-ary tree consisting of uplinks. Up sub-butterfly $\mathrm{BF}^+_{\langle i,\alpha \rangle}(k, m)$ is sub-butterfly that consists of uplinks. The down-tree and down sub-butterfly are defined similarly for downlinks.
Up sub-butterfly $\mathrm{BF}^+_{\langle i,\alpha \rangle}(k, m)$ is the same sub-butterfly as $\mathrm{BF}^-_{\langle i \oplus m,\alpha \rangle}(k, m)$. In few lemmas we will be using both up-tree $T_m^+(v)$ and down-tree $T_m^-(v)$ rooted in the same vertex v. We willtherefore define two-tree $T_m^{\pm}(v)$ as a conjunction of up-tree $T_m^+(v)$ and down-tree $T_m^-(v)$

3.1 Lower bound

We prove that information is not broadcasted in $\theta(n) = \left\lfloor \frac{3}{2}n \right\rfloor$ time steps. In other words Broadcast is not completed in less than $\left\lfloor \frac{3}{2}n \right\rfloor + 1$ time steps. We prove this by showing, that in any path of length $\left\lfloor \frac{3}{2}n \right\rfloor$ from initiator v_0 to v whose distance is $\left\lfloor \frac{3}{2}n \right\rfloor$ there is fault.
Let be v_1 the only informed vertex adjoint to v_0 in time step 1. All other vertices are cut off by fault. In next time step let be all faults on edges leading from v_1 but to v_0. This cuts off any path from v_0 to v leading through v_1. All the other paths were cut off in time step 1. Thus leaving no possibility to have path $\left\lfloor \frac{3}{2}n \right\rfloor$ long.

3.2 Upper bound

In [DV99] there was used idea of isoperimetric number to prove upper bound for hypercubes. It might be tempting to use this idea even on this Cayley graph. However there is one problem. Even if we use $2k$ as our isoperimetric number, which is degree of k-ary wrapped butterfly, in time $\frac{diam(G)}{2} = \frac{3n}{2}$ we have only $(2k)^{\frac{3n}{4}}$ informed vertices. To complete idea next equation needs to be valid.

$$(2k)^{\frac{3n}{4}} > \frac{n}{2}k^n$$

$$2^{\frac{3n}{4}} > \frac{n}{2}\frac{k^n}{k^{\frac{3n}{4}}}$$

$$2^{\frac{3n}{4}} > \frac{n}{2}k^{\frac{n}{4}}$$

But for $k = 8$

$$2^{\frac{3n}{4}} < \frac{n}{2}8^{\frac{n}{4}}$$

$$2^{\frac{3n}{4}} < \frac{n}{2}2^{\frac{3n}{4}}$$

Though we may use another idea used in this paper. If we know that W vertices are be informed in w time steps. We also know that if there is W uninformed vertices, in next time w time steps information is be broadcasted to all vertices.

To find upper bound of message broadcasting within non-wrapped butterfly network we used embedded tree and counted number of informed vertices within last but one level (lemma 2.8). For wrapped butterfly network we run into some problems why we cannot do it this way.
We cannot chose one tree that we will be studying in time step 0, cause root has only k adjoint vertices within this tree and $2k - 1$ faults will ensure that all are faulty. The same problem is if we say we try to choose the tree in time step 1. In this time step we can ensure only 2 vertices being informed and these two vertices have in any tree $2k - 1$ faults.
Choosing tree in time step 2 is one solution. Though by choosing tree in second time step we ignore some vertices and with combination with $2k - 1$ faults, it will lead to badly interpretable and non-optimal results.

Thus for proving upper bound in this paper we are watching all four k-ary Trees rooted in two vertices informed in time step 1. These vertices we denote as $v_0 = \langle 0, \mathbf{0} \rangle$ initiator of broadcast and v_1 vertex that is informed in time step 1 by initiator.

First lemma starts process of broadcasting within all four trees. And counts number of vertices informed in a time step as well as vertices newly informed in the time step.

Lemma 3.1 *In time step t, $1 < t \le \lfloor n/2 \rfloor$ there is at least $k^{t-1} + \sum_i^{t-1} k^i$ informed vertices. And a subset of at least $(2k-1)k^{t-2}$ vertices was newly informed in time step t*

Proof In time step $t = 2$ informed vertices v_0, v_1 have potential to inform $2k-1$ neighbors each. There is maximally $2k-1$ faults leaving us there is at least $2k-1 = (2k-1)k^{2-2}$ newly informed vertices. And at least $2+2k-1 = k^1+k^1+1$ informed vertices totally. All these vertices are in k-ary Tree rooted in v_0 or v_1 respectively, cause all are adjoint to one of v_0, v_1.

In time step $t+1$. We can inform k new vertices from all newly informed vertices in time step t and vertices which was not informed in time step t because of fault. Thus in time step $t+1$ there is $(2k-1)k^{t-2}k + (2k-1) - (2k-1)$ newly informed vertices.
In time step t there are $k^{t-1} + \sum_i^t k^i$. Adding newly informed vertices in $t+1$ we have $k^{t-1} + \sum_i^{t-1} k^i + 2k^t - k^{t-1} = k^t + \sum_i^t k^i$

$\bowtie$

First lemma started process of dissemination of message within four trees, but there is problem after $\lfloor n/2 \rfloor$ time steps. The problem that after this time step, the vertices informed by uplinks and by downlinks might get mixed. How we treat and count these vertices is shown in next lemma.
Lemma 3.3 uses this knowledge and counts the number of informed vertices within time steps after $\lfloor n/2 \rfloor$.

Lemma 3.2 *Let be T be set of vertices informed in time step t. Let be P_i^+ set of vertices which can be informed in $t+i$ time steps by uplinks only. Let be S_i^+ vertices informed in $t+i$ time steps by uplinks only. Let be P_i^- and S_i^- defined equally for downlinks.*
In time step $t+i$ there will be at least $|S_i^+|+|S_i^-|-|P_i^+ \cap P_i^-|$ informed vertices.

Proof If fault in uplink is also fault in downlink, both vertices of the faulty edge are informed, nilling effect of this fault at all. Thus we know there cannot be any fault affecting more vertices than in case of uplink or downlink only. If we know number of vertices informed by uplink without considering downlink $|S_i^+|$ and number of vertices informed by downlink without considering uplink $|S_i^-|$ we know, there will be at least $|S_i^+ \cup S_i^-| = |S_i^+|+|S_i^-|-|S_i^+ \cap S_i^-|$ informed vertices total. Set of vertices $S_i^+ \cap S_i^-$ is subset of intersection $P_i^+ \cap P_i^-$. Cause we can evaluate $|P_i^+ \cap P_i^-|$ for any possible combination of faults. We use $|S_i^+| + |S_i^-| - |P_i^+ \cap P_i^-| \le |S_i^+ \cup S_i^-|$ as the estimation of all informed vertices.

$\bowtie$

Lemma 3.3 *In time step $t+1$, where $t < n$ there is at least $k^{t-1} + \sum_i^{t-1} k^i - kn$ informed vertices.*

Proof By Lemma 3.2 we know that we can count number of informed vertices in $BF(k,n)$ by the same algorithm used in in Lemma 3.1 with subtracting the possible intersection. vertices potentially informed in time step t by uplink are subset of all vertices potentially informed by uplink in time step n, $P^+_{n-1} = T^+_{n-1}(v_0) \cup T^+_{n-1}(v_1)$. For downlink equally $P^-_{n-1} = T^-_{n-1}(v_0) \cup T^-_{n-1}(v_1)$.
For describing intersection we will look at each level i separately. We will use $\mathbf{K} = \{0, \ldots, k-1\}$ and $v_1 = \langle 1, \mathbf{v_1} 0^{n-1} \rangle$.
For $i > 1$:
$T^+_{n-1}(v_0) \cap T^-_{n-1}(v_0) = \{\langle i, \mathbf{K}^i 0^{n-i} \rangle\} \cap \{\langle i, 00^{i-1}\mathbf{K}^{n-i} \rangle\} = \langle i, 0^n \rangle$
$T^+_{n-1}(v_0) \cap T^-_{n-1}(v_1) = \langle i, \mathbf{K}^i 0^{n-i} \rangle \cap \langle i, \mathbf{K}0^{i-1}\mathbf{K}^{n-i} \rangle = \langle i, \mathbf{K}0^{n-1} \rangle$
$T^+_{n-1}(v_1) \cap T^-_{n-1}(v_0) = \langle i, \mathbf{v_1}\mathbf{K}^{i-1} 0^{n-i} \rangle \cap \langle i, 00^{i-1}\mathbf{K}^{n-i} \rangle = \emptyset$
$T^+_{n-1}(v_1) \cap T^-_{n-1}(v_1) = \langle i, \mathbf{v_1}\mathbf{K}^{i-1} 0^{n-i} \rangle \cap \langle i, \mathbf{K}0^{i-1}\mathbf{K}^{n-i} \rangle = \langle i, \mathbf{v_1} 0^{n-1} \rangle$
We also know that $P^+_{n-1}(P^-_{n-1})$ has only k vertices in level 1 (0). Thus in all levels there is at most k vertices within intersection, this means $|P^+_i \cap P^-_i| = kn$. Using Lemma 3.2 we have in time step $t \leq n$ there is at least $k^{t-1} + \sum_i^{t-1} k^i - kn$ informed vertices.

⋈

We proved that we can continue with the process of message dissemination described in Lemma 3.1 for i larger than $\lfloor n/2 \rfloor$. But there emerge another problem in this process, for small wrapped butterflies (butterflies with small n). By Lemma 3.2 we know that the intersection of up-trees and down-trees needed to be subtracted from number of informed vertices. Within butterfly with small n, this intersection becomes to high portion of informed vertices. Thus at first we will prove that we can broadcast message in less optimal time $4n-2$ for any combination of n's and k's.

Lemma 3.4 *In $4n-2$ time steps whole BF(k,n) will be informed.*

Proof In time step $i < n$ there is informed path of length $i+1$ within butterfly, with one vertex per level. In time step $i+1$ there are at most $(2k-1)$ faults but there are $2k$ vertices adjoint to outermost vertices of the path that can lengthen it. Thus in any time step $i < n$ there is informed path of length $i+1$. In time step $n-1$ there is informed vertex within each level.
Now we will watch progress of information in all trees T_n rooted in these n vertices. Particularly we watch leafs of these trees. Leafs compose whole wrapped butterfly and each vertex might be informed by up-Tree or down-Tree. By Lemma 3.2 we know that we can ignore any interference from opposite directions if we don't forget final intersection.

In time step $n-1+i$ there might be $2k-1$ faults. Each fault will prevent k^{n-i} vertices from being informed. In n time steps cumulatively it is $\sum_{i=1}^{n}(2k-1)k^{n-i} > k^n + \sum_{i=0}^{n} k^i$

In time step $n-1+n$ the message is broadcasted to $2nk^n - k^n - \sum_{i=0}^{n} k^i$ non distinct leaf vertices. Each leaf vertex might be informed by up-link or by down-link, thus there is at least $\frac{1}{2}\left(2nk^n - k^n - \sum_{i=0}^{n} k^i\right)$ informed vertices.

$$
\begin{aligned}
\frac{1}{2}\left(2nk^n - k^n - \sum_{i=0}^{n} k^i\right) &> \frac{n}{2}k^n \\
2nk^n - k^n - \sum_{i=0}^{n} k^i &> nk^n \\
2nk^n - k^n - k^n - nk^n &> \sum_{i=0}^{n-1} k^i \\
(n-2)k^n &> \frac{k^n-1}{k-1} \\
(n-2)k^n &> k^n - 1 \qquad\qquad (k \geq 2) \\
n-2 &> \frac{k^n-1}{k^n} \\
n &> 2 + \frac{k^n-1}{k^n} \\
n &\geq 3
\end{aligned}
$$

By definition of wrapped butterfly network we know that $n \geq 3$ thus in $2n-1$ time steps there are more than $\frac{|V|}{2}$ vertices informed.

In the beginning of this section we noted that time to inform last $\frac{|V|}{2}$ is the same as informing first $\frac{|V|}{2}$ vertices. Thus the information within remaining $\frac{|V|}{2}$ vertices will be broadcasted in at least $2n-1$ time steps. Leaving $4n-2$ as total broadcast time.

$\bowtie$

We can split the process of broadcasting message within wrapped Butterfly into two distinct parts. First part is broadcasting the message to $\Theta(k^n)$ vertices in one level and second part broadcasting message by level by level to all vertices. For the intermediate state we now chose state where all sub-butterflies $\mathrm{BF}^{+}_{\langle 0,\alpha\rangle}(k,s)$ (and $\mathrm{BF}^{-}_{\langle 0,\alpha\rangle}(k,s)$) have informed level. This position can be achieved by informing all vertices in level 0 within some sub-butterfly $\mathrm{BF}^{+}_{\langle s,\beta\rangle}(k,n-s)$.

Next lemma searches for preconditions that need to be met for finding such a sub-butterfly.

Lemma 3.5 *Let us denote $s = \lfloor \log_k n \rfloor + 3$. If $s < n$ and there is $T_s\langle 0, \mathbf{0}\rangle$ informed in time step t all sub-butterflies $BF_{\langle 0,\alpha\rangle}(k, s)$ are going to have informed vertex in time step $t + n - s$.*

Proof There exists k^s vertex-disjoint paths between any $\mathrm{BF}_{\langle 0,\alpha\rangle}$ and $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}$. End vertices of these paths within $\mathrm{BF}_{\langle 0,\mathbf{0}\rangle}$ are leaf vertices of Tree $T_s\langle 0, \mathbf{0}\rangle$ We know that there exists k^s vertex-disjoint paths between each $\mathrm{BF}_{\langle 0,\alpha\rangle}(k, s)$ and informed leaf within T_s.
We also know that there are at most $(n - s)(2k - 1)$ faults within time steps $t+1, \ldots, t+n-s$. And $(n-s)(2k-1) < n.2.k < k^{\log_k n}.k^2 < k^{\lfloor \log_k n\rfloor+3} = k^s$ thus there is more paths between informed vertex and $\mathrm{BF}_{\langle 0,\alpha\rangle}(k, s)$ for any $\langle 0, \alpha\rangle$. Thus in time step $t + n - s$ there is informed vertex within each $\mathrm{BF}_{\langle 0,\alpha\rangle}(k, s)$.

$\bowtie$

In last lemma we proved that for finding sub-butterfly with informed vertices in its last level we need to find completely informed tree T_s. z
Next lemma will tell us what is the time needed to find such Trees.

Lemma 3.6 *For s, where $s + \frac{3}{2}\log_k s + 2 < \frac{n}{2}$ and $\lfloor \log_k 2s\rfloor < \frac{n}{2}$, there exists fully informed tree $T_{2s}\langle i, \alpha\rangle$ in time step $2s + \lfloor \log_k 2s\rfloor + 2$.*
In time step $2s + \lfloor \log_k 2s\rfloor + 2$ there exist fully informed trees $T_s^+\langle i + s, \alpha\rangle$ and $T_s^-\langle i + s, \alpha\rangle$.

Proof Let be $S = \lfloor \log_k 2s\rfloor + 2$ By Lemma 3.1 we know that in time step S there is at least $(2k - 1)k^{S-1}$ newly informed vertices. We also know that in next $2s$ time steps there will be at most $(2k - 1)2s$ faults.

$$(2k - 1)2s < (2k - 1)k^{\lfloor \log_k 2s\rfloor+1} = (2k - 1)k^{S-1}$$

From this follows, there are more informed vertices in time step S than possible faults in next $2s$ time steps. Thus in time steps $S, \ldots, S + 2s$, there exists vertex $v = \langle i, \alpha\rangle$ such that all information dissemination behind level S by this vertex v, is fault-safe in time steps $S, \ldots, S + 2s$. From which directly follows that in time step $S + 2s$, vertex v is root of completely informed tree $T_{2s}(v)$. Without loss of generality we will assume that vertex v is informed by uplinks and $T_{2s}(v) = T^+_{2s}(v)$
For vertex $v' = \langle i + s, \alpha\rangle$ we know that it is informed in time step $S + s$ and Tree $T_s^+(v') \subset T^+_{2s}(v)$ is in fail-safe area of vertex v in time steps $S+s, \ldots, S+2s$. But also Tree $T_s^-(v')$ is in fail-safe area of vertex v in time steps $S+s, \ldots, S+2s$, cause whole down-Tree $T_s^-(v')$ is in levels greater or equal to S.
There exists fully informed $BF_{\langle i,\alpha\rangle}(k, s)$, but we won't be using this knowledge. By Lemma 3.2 uplinks and downlinks behind levels S and $-S$ won't interfere,

leaving condition $2S + 2s < n$ for $T_{2s}(v)$. $\lfloor \log_k 2s \rfloor + 2 + s < \frac{n}{2}$
By Lemma 3.2 uplinks and downlinks behind levels S and $-S$ can interfere for $T_s^{\pm}(v')$. But behind levels $S + s$ or $-(S + s)$ all is safe again, leaving condition $2S + 3s < n$ for $T_s^{\pm}(v')$. $\lfloor \log_k 2s \rfloor + 2 + \frac{3}{2}s < \frac{n}{2}$

$\bowtie$

Further we will use two-tree $T^{\pm}(s)$ to broadcast message to all $BF^{+}(k,s)$ and $BF^{-}(k,s)$ beginning in the same level as v. We had to at first compute s. This we can manage with knowledge that in $(n - s)$ time steps there will be at most $(2k-1)(n-s)$ faults in $(n - s)$ time steps. and in we have $2k^s$ informed vertices.k

Lemma 3.7 *Let us denote* $s = \lfloor \log_k n \rfloor + 3$. *If* $s + \frac{3}{2}\log_k s + 2$ *then in time step* $n + \lfloor \log_k n \rfloor + log_k 2s + 5$ *there is informed vertex within level* 0 *in each* $BF^{+}_{\langle 0,\alpha\rangle}(k,s)$ *and* $BF^{-}_{\langle 0,\alpha\rangle}(k,s)$.

Proof From Lemma 3.6 we know that in time $2s + \log_k 2s + 2$ there exist vertex $\langle i,\alpha\rangle$ such that $T_s^{+}(\langle i,\alpha\rangle)$ and $T_s^{-}(\langle i,\alpha\rangle)$ are informed. Without loss of generality we will denote $\langle i,\alpha\rangle$ as $\langle 0,\mathbf{0}\rangle$.
By Lemma 3.2 we know that we can broadcast information simultaneously by uplinks and downlinks without interference. Thus by using Lemma 3.5 for up-Tree an down-Tree separately we get informed vertex in each $\mathrm{BF}^{+}_{\langle 0,\alpha\rangle}(k,s)$ and $\mathrm{BF}^{-}_{\langle 0,\alpha\rangle}(k,s)$ in time step $2s + \log_k 2s + 2 + n - s = n + s + \log_k 2s + 2$

Lemma 3.8 *In* $\left\lfloor \frac{3}{2}n \right\rfloor + O(\log_k n) = \left\lfloor \frac{3}{2}n \right\rfloor + 10\log_k n + 20$ *time steps whole Butterfly* BF(k,n) *is informed.*

Proof Let $s = \lfloor \log_k n \rfloor + 3$ By Lemma 3.7 we know there is informed vertex in each $\mathrm{BF}_{\langle 0,\alpha\rangle}(k,s)$ in both directions in time step $T = n + \lfloor \log_k n \rfloor + log_k 2s + 5$. In process of broadcasting we will cut Butterfly into smaller parts first part are $\mathrm{BF}_{\langle 0,\alpha\rangle}(k,s)$ which have informed vertex. We count next part as smallest butterflies that have at least one informed vertex.
Let $s_0 = s$. Let s_{i+1} be defined as $s_{i+1} = \lfloor \log_k s_i \rfloor + 3$. Each $\mathrm{BF}(k, s_{i+1})$ is connected with $k^{s_{i+1}}$ $\mathrm{BF}(k,s_i)$'s through exactly one vertex. We know that there will be at most $(2k-1)s_i$ faults while broadcasting throughout $\mathrm{BF}(k,s_i)$. But $(2k-1)s_i \leq k^2 k^{\log_k s_i} \leq k^{s_{i+1}}$ thus there is more paths between informed level in time step $T + \sum_{i=0}^{i-1} s_i$ and $\mathrm{BF}(k,s_{i+1})$'s than possible faults in time steps $T + (\sum_{j=0}^{i-1} s_j) + 1, \ldots, T + (\sum_{j=0}^{i} s_j)$.
Sequence of s_i's has fix point value for every k let us denote this value as $s(k)$. Let be n' such that $\sum_{j=0}^{n'} s_j \leq n < \sum_{j=0}^{n'+1} s_j$. For any n we prove that $s_{n'} \leq s(k) + 3$ and $s_{n'+1} = s(k) + 1$.

If $n' = 0$ (contradiction) Let us assume $s_0 = s(k) + q$ where $q > 3$ and count smallest possible n. From $s(k) = \lfloor \log_k s(k) + 3 \rfloor$ we know $k^{s(k)} = k^{\lfloor \log_k s(k) \rfloor + 3} \geq k^{\log_k n + 2} = s(k).k^2$. Let us have a look at what need to n in case $s_0 = s(k) + q$

$$n \geq k^{\lfloor \log_k n \rfloor} \geq k^{s_0 - 3} = k^{s(k)+q-3} \geq s(k)k^{q+2-3} = s(k)k^{q-1}$$

From this we know that $s_0 = s(k) + q$ there needs to be n at least $s(k)k^{q-1}$. For $q > 3$ it is at least $s(k)k^3$

$$s(k)k^{q-1} \leq n$$
$$s(k)k^{q-2} \leq \frac{n}{k}$$
$$(s(k)k^{q-3})k \leq \frac{n}{2}$$
$$(s(k) + q)k \leq \frac{n}{2}$$
$$2s_0 \leq \frac{n}{2}$$

But if $2s_0 \leq \frac{n}{2}$ there is $s_1 = \lfloor \log_k s_0 \rfloor + 3 \leq s_0$ and $s_0 + s_1 \leq \lfloor \frac{n}{2} \rfloor$ which contradicts to assumption that $n' = 0$.
If $n' > 0$ the process is equal but error will be revealed in $s_{n'-1}$ and escalated to n.
If $s_{n'} \leq s(k) + 3$ then $s_{n'+1} = \lfloor \log_k s_{n'} \rfloor + 3 \leq \lfloor \log_k (s(k) + 3) \rfloor + 3 \leq s(k) + 1$

Thus for any n satisfying our initial assumption, If $\sum_{i=0}^{n'+1} s_i > \lfloor \frac{n}{2} \rfloor$ and $\sum_{i=0}^{n'} s_i \leq \lfloor \frac{n}{2} \rfloor$ than $s_{n'} \leq s(k) + 1$
In $T + \lfloor \frac{n}{2} \rfloor + s(k) + 1$ time steps we have message broadcasted through half of Butterfly with Partitioning to BF(k, s_i)'s. Also In opposite direction information was broadcasted in half of Butterfly, thus we have whole Butterfly partitioned to Butterflies BF(k, s_i). This partition butterflies had informed vertex at the start of transmitting message through it. For each BF(k, s_i) there was also s_i time steps from obtaining first informed vertex. After these additional s_i time steps there will be informed at least $k^n - (2k-1)\sum_{j=0}^{i} k^j$ vertices inside level $\sum_{j=0}^{i} s_j$.
From $s_i \geq s_i + 1$ we know that in next s_0 time steps all of these informed vertices will can broadcast its message to level $\sum_{j=0}^{i-1} s_j$. Thus in time step $T + \lfloor \frac{n}{2} \rfloor + s(k) + 1 + s_0$ the number of informed vertices will be:

$$2\sum_{i=0}^{n'} s_i(k^n - (2k-1)\sum_{j=0}^{s_i} k^j) - (2k-1)s_0$$

We know $2\sum_{i=0}^{n'} s_i \leq n$, $\frac{n}{s(k)} > n'$ and $s_i \leq s_0 = s$. Thus the number of informed vertices will be at least:

$$nk^n - \sum_{i=0}^{n'}(s(2k-1)\sum_{j=0}^{s} k^j) - (2k-1)s$$

$$
\begin{aligned}
& nk^n - \sum_{i=0}^{\frac{n}{s(k)}} (s(2k-1)\sum_{j=0}^{s} k^j) - (2k-1)s \\
& \geq nk^n - \frac{n}{s(k)}(s(2k-1)sk^s) - (2k-1)s \\
& \geq nk^n - (2k-1)\left(\frac{ns^2}{s(k)}k^s + s\right) \\
& \geq nk^n - k^2k^s\left(\frac{ns^2}{s(k)} + o(1)\right) \\
& \geq nk^n - k^2k^s(n^3) \\
& \geq nk^n - k^{s+2+3\lfloor \log_k(n) \rfloor+3} \\
& \geq nk^n - k^{4s-4}
\end{aligned}
$$

We know that whole butterfly $\mathrm{BF}(k,n)$ is informed in time equal to time needed to broadcast message across k^{4s-4} vertices. By Lemma 3.6 we know that in $4s-3$ time steps, where $4s-3 < n$, there are $k^{4s-3} - kn \geq k^{4s-4} + (k-1)k^{4s-4} - k^{s-3} \geq k^{4s-4}$ vertices informed. Thus whole $\mathrm{BF}(k,n)$ will be informed in time step $T + \lfloor \frac{n}{2} \rfloor + s(k) + 1 + s + 4s - 3 = \lfloor \frac{3}{2}n \rfloor + 6s + \log_k 2s + s(k) = \lfloor \frac{3}{2}n \rfloor + 6\lfloor \log_k n \rfloor + \log_k(2\lfloor log_k n \rfloor) + s(k) + 18$.

For butterflies where $4s-3 \geq n$ we can use Lemma 3.4 and for $4n-2 = \frac{3}{2}n + \frac{5}{2}n - 2 \leq \frac{3}{2}n + 10s - 9.5$

Finally we know that message will be broadcasted in at least $\frac{3}{2}n + 10s$ or $\frac{3}{2}n + 10\log n + 20$

⋈

Lemma 3.9 [2] *Broadcast is completed in* $\frac{3}{2}n + s + \log_k 2s + O(s(k))$ *time steps. Where* $s(k) = O(1)$ *and* $s = \lfloor \log_k n + 3 \rfloor$.

Proof We will continue with process from previous lemma in point where we defined sequence $s_0, \ldots, s_{n'}$. Let us denote n'' lowest index such that $s_{n''} = s(k)$. For beginning we will assume that $\sum_{i=0}^{n''} \leq \frac{n}{4}$.
Thus we have in time step $n + \lfloor \log_k n \rfloor + log_k 2s + O(1) + \frac{n}{4}$ informed sub-butterflies $\mathrm{BF}(k, s(k))$ for levels around level $\frac{n}{4}$. In O(s(k)) time steps all but two sub-butterflies have to be informed. Now in next $\frac{n}{4} + O(1)$ time steps all vertices will be informed. For idea on proving time constrains of level-by-level broadcasting see attached Lemma 6.1
In case that $\sum_{i=0}^{n''} > \frac{n}{4}$. We know that $n = O(s(k))$. (Exactly $n < 17s(k)$ which is needed for worst combination $n = 80, k = 2$ where $s_0 + s_1 + s_2 = 9+6+5 = 20$). $4n-2$ time steps from lemma 3.4 can be rewritten into $\frac{3}{2}n + O(s(k))$.

⋈

[2] Results of this lemma does not improve $diam(G) + O(\log_k n)$ but is usefull for showing other idea. That is reason why it is not proved exactly.

4 Conclusion

In the introduction of this thesis we stated questions. Now we have knowledge to give an answers. Is butterfly network topology good fault tolerant network when dealing with dynamic faults in broadcasting? In particular butterfly networks are not good for faulty network, cause even one dynamic fault can blow up broadcasting time by 50%. This problem can be circumvented by wrapping butterfly. Butterfly networks do not have this particular problem, cause information can be disseminated in two directions. Also by wrapping butterfly we doubled number of faults that network can resist. This also means that there is more interference by more faults. We proved (Lema 3.8) that the interference causes not more than logarithmic (to n) slowdown. (Or twice logarithmed number of vertices in network).

In comparison to known results 4 we see that Butterfly network is worse than hypercubes even when comparing these not so optimal results of lower bound. In case of wrapped Butterfly networks situation is not so straightforward. Lower bound is 50% better and upper boud 50% worse than that of hypercube.

Let us hypothesize about final results that might be proved for wrapped butterflies. For upper bound there is space to improve whole first part, informing all sub-butterflies. Probably in a way that finding one sub-butterfly in which no fault can occur we won't search by using trees. This might save us $s + \log_k 2s$ time steps pending in all our results. In such case we will have $diam(BF(k, n)) + O(s(k))$ time steps if we just add second part of this proof. Proving thus that $diam(BF(k, n)) + O(1))$ time steps are sufficient. Another problem with optimizing results there is that for lower n and k broadcast can be slowed more than for higher ones. Thus resolving a descening function is needed. In this example we used $s(k)$ which will never be lower than 3 thus for big n presented procedure cannot lead to upper bound better than $diam(BF(k, n)) + 3$. After resolving this apropriate descending function there will be space to improve lower bound also.

deg	Wrapped Butterfly		Hyper-cube	Butterfly	
	(lower)	(upper)		(lower)	(upper)
2	25	62	-	48	70
3	17	42	-	33	46
4	14	34	-	27	37
10	10	22	-	18	24
20	8	18	12	15	20
50	7	14	-	12	17
100	5	10	-	9	13

Table 3: Graph broadcast time for $|V| = 10^6$.

5 Bibliography

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6 Appendix

This lemma was developed while researching possible broadcast procedures. It proof consists of too many cases for little overall gain, thus it was removed from main prcedure.

Lemma 6.1 *Let the level l in wBF(k,n) where $k > 2$ be completely informed in time step t. In time at least $t + \left\lfloor \frac{n}{2} \right\rfloor$ there will be at most three uninformed vertices in at most two levels of wBF(k,n).*

Proof Now we have to notify that there are only three different cases three uninformed vertices could be arranged, in time step i.
Case 1:

At most two vertices in levels $l \pm i_1$ and $l \pm i_2$, where $i - i_1 \leq 1$ and $i - i_2 \leq 1$)
Case 2:

$k = 4$,$k = 5$ Three vertices in the same level $l \pm i$ and all three vertices are in the same sub-butterfly $\text{BF}_{\langle i-1,\alpha\rangle}(k, 1)$ for some α.

$k = 3$ two of three vertices are in the same level $l \pm i$. and in the same sub-butterfly $\text{BF}_{\langle i-1,\alpha\rangle}(k, 1)$ for some α.
Case 3:

One vertex at level $l + i - 2$ $(l - i + 2)$
First of all we prove that in time step $t + 1$ we will have one of this cases. In time step $t + 1$ vertex in levels $l - 1$, $l + 1$ is not informed in case that there are k faulty channels from level l to this edge. In one time step there is at most $2k - 1$ faulty channel. Thus in time step $t + 1$ there is only one uninformed vertex in levels $l - 1, l + 1$. Case 1.
In time step $t + i$.
Case 1.

For this case we have three subcases
a. $i_1 = i_2 = i - 1$ Both uninformed vertices have $2k$ informed neighbour vertices thus will be informed in time step $t + i + 1$. We now have informed levels $t + i$ and $t - i$. The same situation as in time step t and level l. Case 1.
b. $i_1 + 1 = i_2 = i$ We now have two uninformed vertices one on level $l + i - 1$ with at least $2k - 1$ adjoint informed vertices. One on level $l + i$ with $k - 1$ adjoint informed vertices.
If the uninformed vertex in level $l + i - 1$ will stay uninformed in time step $t + i + 1$ there will be no other uninformed vertices in levels $l - i - 1, \ldots, l + i + 1$. cause there is only $2k - 1$ possible faults, which are needed for isolating this vertex. Case 1.
If all vertices will be informed in level $l + i - 1$ in time step $t + i + 1$ there are only vertices with $k - 1$ or k informed neighbours in levels $l + i$,$l + i - 1$. Thus in time step $t + i + 1$ there will be at most two uninformed vertices in levels $t + i$, $t + i + 1$. Case 1.

c. $i_1 = i_2 = i$
Let us consider this two uninformed vertices are not in $\mathrm{BF}_{\langle i,\alpha\rangle}(k,1)$. for any α. All vertices will have at least $k-1$ informed adjoint vertices. Thus two vertices might be uninformed. Case 1.
Now let us consider both of these vertices are in one $\mathrm{BF}_{\langle i,\alpha\rangle}(k,1)$. Then there exists k vertices in level $l+i+1$ adjoint only to $k-2$ informed vertices.

$k=3$ Vertices in $\mathrm{BF}_{\langle i,\alpha\rangle}(k,1)$ If there are three uninformed vertices in time step $t+i+1$ there had to be two of them within $\mathrm{BF}_{\langle i,\alpha\rangle}(k,1)$ $(2(k-2)+k \leq 2k-1$ but $2k-1 < k-2+2k)$. The third vertex could be any uninformed vertex $(2(k-2) = 2k-4 = k-1$ with $2k-1 = k-1+k$ faults, there would be k spare faults). Case 1,2.

$k=4$ or $k=5$ If there are three uninformed vertices in time step $t+i+1$ they had to be within $\mathrm{BF}_{\langle i,\alpha\rangle}(k,1)$ $(3(k-2) \leq 2k-1 < 2(k-2)+k)$ and all have to be within level $l+i+1$. Cases 1,2.

$k>5$ There cannot be three uninformed vertices at time step bigger than t. Even in situation where three vertices have $k-2$ uninformed adjoint vertices there are not enough faults for al of tem $(3(k-2) = (2k-1)+(k-5) > 2k-1)$. Case 2.

$k>3$ All uninformed vertices in time step $t+i$ are on the same level. We know that there are k informed adjoint vertices for each uninformed vertex. From information all vertices are in the same sub-butterfly $\mathrm{BF}_{\langle i-1,\alpha\rangle}(k,1)$ we know that no two vertices are in the same sub-butterfly $\mathrm{BF}_{\langle i,\alpha'\rangle}$(k,1). Thus no vertex within levels $l-1, l+1$ has two adjoint vertices uninformed in time step $t+i$.So we know that vertices on levels $l-i-1,\dots,l+i+1$ are adjoint to either k or $k-1$ informed vertices. From that we have that only two vertices from them will be uninformed in time step $t+i+1$. Which is case 1.

$k=3$
Case 3.

Uninformed vertex has $2k$ different informed neighbours, thus will be informed in time $t+i+1$. We now have informed levels $t+i$ and $t-i$ the same as time step t and level l. Case 2.

$\bowtie$

Abstrakt

Na rozdiel od *lokalizovaných* zlyhaní komunikácie, ktoré sa vyskytujú na vopred určenej (a teda známej) množine liniek, *dynamické* zlyhania sa môžu vyskytovat' na l'ubovolnej linke. Takéto zlyhania sú taktiež známe pod názvami mobilné alebo všadeprítomné. Ich prítomnost' st'ažuje niekedy aj znemožňuje riešenie problémov i v synchrónnych systémoch. Preto budeme analyzovat' rozposielanie správ s dynamickými zlyhaniami v synchrónnom komunikačnom móde známom ako shouting mode. V shouting mode môže každý vrchol v sieti informovat' všetkých svojich susedov v jednom časovom kroku. V každom kroku sa môže vyskytovat' nanajvýš tol'ko zlyhaní kol'ko je hranová súvislost' siete. Problémom je nájst' horné i dolné ohraničenie počtu časvoých krokov, ktoré sú potrebné na ukončenie rozposielania správy.

Dokážeme, že pre k-árny butterfly network dimenzie n je spodným ohraničením $3n$ a horným ohraničením $3n + O(\log_k n)$.

Taktiež dokážeme, že pre k-árny wrapped butterfly network dimenzie n je spodným ohraničením $\frac{3}{2}n + 1$ a horným ohraničením $\frac{3}{2}n + O(\log_k n)$.