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On Disjunction in Modal Logics

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I hereby declare that this thesis is my own work, only with the help of the referenced literature and under the careful supervision of my thesis advisors.

Bratislava. May 7, 2007 Peter Drábik

Acknowledgments

– *No, come on, come on, tell me why.*

– *It's like a fellow I once knew in El Paso. One day, he just took all his clothes off and jumped in a mess of cactus. I asked him that same question, "Why?"*

– *And?*

– *He said, "It seemed to be a good idea at the time."*

(The Magnificent Seven)

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Abstract

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This master's thesis consists of two parts.

The first part is a survey of basic concepts in modal logics. Syntax and semantics for normal and non-normal modal logics are presented. Then soundness and completeness theory and correspondence theory of normal modal logic are presented.

The second part is devoted to investigation of distribution of the modal operator over disjunction, i.e. exploring the properties of formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ in both normal and non-normal modal logics. A class of frames that is defined by this formula (the class of deterministic frames) is characterized. Soundness and strong completeness of the smallest normal modal logic containing this formula with respect to the class of deterministic frames is established. A class of neighbourhood frames defined by this formulas is characterized by the property of non-emergence. Testing of non-emergence of neighbouhoods is investigated.

Keywords: modal logic, non-normal, disjunction, distribution, neighbourhood, non-emergence

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Notation

Symbol	*Meaning*	*Definition*
Φ	set of all propositional variables	section 2.1
Φ^{PL}	set of all propositional formulas	section 2.1
Φ^{ML}	set of all modal formulas	section 2.1
$\Box$	modal operator "box"	section 2.1
$\Diamond$	modal operator "diamond"	section 2.1
$p, q, p_i, \ldots$	propositional variables	section 2.1
$\phi, \psi, \phi_i, \ldots$	formulas	section 2.1
Φ^*	set of all quasi-atomic formulas	section 2.2
Σ	modal logic	section 2.2
RPL	rule of propositional logic	section 2.2
PL	set of all tautologies, propositional logic	section 2.2
K,T,B...	schemas	section 2.2.1
$K, KT, KB \ldots$	normal modal logics	section 2.2.1
RN	rule of necessitation	section 2.2.1
RE	rule of equivalence	section 2.2.2
$w, v, v_i, \ldots$	states/worlds	section 2.3.1
$W, A, B, \ldots$	sets of worlds	section 2.3.1
$\mathcal{F}$	standard frame	section 2.3.1
$\mathcal{M}$	standard model	section 2.3.1
C	class of frames	section 2.3.1
$\mathfrak{F}$	neighbourhood frame	section 2.3.2
$\mathfrak{M}$	neighbourhood model	section 2.3.2
N	neighbourhood	section 2.3.2
$\mathbb{N}$	set of all natural numbers	–
Σ_C	logic of C	section 2.4.1
Δ	set of formulas	section 2.4.2
$\Sigma \Vdash^S \phi$	local consequence	section 2.4.2
$\mathcal{M}^\Sigma = (W^\Sigma, R^\Sigma, V^\Sigma)$	canonical model	section 2.4.2
$Distr\vee$	formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$	chapter 3

Chapter 1

Introduction

1.1 Modal logic

Modal logic is a broad research area enjoying much attention in the past few decades.

Modal logic is a formal system used for handling modalities: concepts like possibility, existence, and necessity. Logics for handling a number of other ideas, such as eventually, formerly, can, could, might, may, must are by extension also called modal logics, since it turns out that these can be treated in similar ways. [14]

Founded by C. I. Lewis in the beginning of 20th century, modal logic was essentially considered a logic of necessity and possibility. Since 1959, when American philosopher and logician Saul Kripke introduced the semantics for modal logics, we have been talking about modern era in modal logic. This semantics, now commonly referred to as Kripke semantics, shifted the meaning of modal logic to being also powerful description language of relational structures.

Modal logic is a branch of science with interesting applications in various fields including philosophy, linguistics, game theory, theoretical computer science, knowledge representation and artificial intelligence. However, the theory itself is at least as interesting as its applications.

The term "modal logic" is used in two meanings in this text. First, in the sense mentioned above, as a scientific discipline studying and using logic with modal operators. Secondly, a modal logic (or a system of modal logic) is a set of formulas constructed in a language with modal operators.

Normal modal logics is a special class of modal logics, those having nice semantical counterparts – standard frames. Non-normal modal logics, those which aren't normal, have deserved less attention of modal logicians so far. Yet, some applications need expressive power lower than that of normal modal logics, which implies a need of studying non-normal logics in depth too.

1.2 Topic and motivation of this work

The main topic of this work is the issue of how the connectives interact in either standard semantics or neighborhood semantics. Motivation for investigating this topic comes from philosophy and logic.

Some properties of normal modal logics can be considered too strong for many applications. Under some interpretations axioms (formulas) correspond to principles philosophically unacceptable for domains they are supposed to describe and it is natural to think that they don't hold.

For instance, often researchers consider the principle of addivity (C) provided by every normal modal logic as inappropriate and want to abandon it. At the same time, they want to retain the converse principle, monotony (M) – also a theorem of every normal modal logic

$$\begin{array}{ll} \text{C.} & (\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi) \\ \text{M.} & \Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi). \end{array}$$

These reasons make researchers abandon normal modal logics and adopt a new framework – non-normal modal logics, where these axioms don't hold generally.

Many recently explored, and independently motivated formalisms treat these principles this way. Concrete examples are Parikh's Game Logic [45], Pauly's Coalition Logic [8] and Alternating-time temporal logic [9].

Recent research using non-normal logics approach has been done in logics of knowledge and belief, where modal operator $\Box$ is adopted as a epistemical operator. Arló Costa has also explored logics of probabilities (see [4], [5], [6]).

This research is not purely philosphically-driven, modal logics are used for describing and reasoning about domains of artificial intelligence and multiagent systems and other application fields.

In this context it is important to see when modal operator interacts with other connectives.

We are interested in how modal operator ($\Box$, so called "box") distributes across connectives in modal logics. This means we ask which modal logics contain schemas of the form

$$\Box(\phi \otimes \psi) \rightarrow (\Box\phi \otimes \Box\psi),$$

where $\otimes$ is any connective.

In normal modal logics, the situation is following. Distribution over conjunction, implication and logical equivalence works, i.e. formulas of that form are valid in any normal modal logic. However, distribution over disjunction doesn't work.

When turning to non-normal logics, with more expressive framework we lose some properties. None of these formulas is valid in all non-normal logics. Distribution of box over implication is an axiom K, defining normal modal logics. Distribution over conjunction is already mentioned schema M. This has been investigated in the literature (see [1]). Distribution over disjunction is what we devote our work to in the next chapters

$$\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi).$$

This formula is called distribution of the modal operator over disjunction. It disassembles a complex modal formula into a boolean combination of less complex formulas. The name comes from the analogue: the way multiplication is distributed over addition in simple algebra.

1.3 Outline of the work

In the next chapter [2] we introduce the basic concepts in modal logics, both syntactical an semantical. We will also provide the reader with necessary soundness and completeness theory and correspondence theory.

Chapter [3] will consist of our results in distribution of modal operator over disjunction in both normal and non-normal logics.

In the last chapter [4] we will summarize the results of this work and sketch the possibilities of further research.

Chapter 2

Basic concepts

This chapter provides basic terms and concepts in modal logic necessary for understanding the following chapters. For more detailed discussion consult [1] and [2].

2.1 Syntax – language

We introduce a language, over which the manipulated objects will be constructed. First step is the language of propositional logic.

Let $\Phi = \{p, q, p_1, p_2, \ldots\}$ be a denumerable set of *propositional variables* and $\neg$ and $\vee$ be the *primitive Boolean connectives*. The logical connectives $\wedge, \rightarrow$ and $\leftrightarrow$ are *derived connectives* defined as abbreviations $\phi \wedge \psi \equiv \neg(\neg\phi \vee \neg\psi)$, $\phi \rightarrow \psi \equiv (\neg\phi \vee \psi)$ and $\phi \leftrightarrow \psi \equiv (\phi \rightarrow \phi) \wedge (\psi \rightarrow \phi)$. Let $\top$ stand for $p \vee \neg p$ and $\bot$ stand for $\neg\top$. The set of *propositional formulas* is the set Φ closed under the primitive Boolean connectives, denoted by Φ^{PL}.

We augment the language by an unary *modal operator* "box": $\Box$. We restrict ourselves to language with only one unary modal operator, i.e. to the basic modal language (see [2]). Then there is one derived modal operator $\Diamond$, "diamond", defined as $\Diamond\phi \equiv \neg\Box\neg\phi$. We say $\Box$ and $\Diamond$ are *duals* of each other.

In this text we consider the propositional case of modal logic, that means we make no use of quantifiers like $\forall$ or $\exists$.

Formally, the set of *modal formulas* Φ^{ML} is defined as follows.

Definition 2.1.1.

1. p is a modal formula if $p \in \Phi$
2. $\top$ is a modal formula
3. $\bot$ is a modal formula
4. $\neg\phi$ is a modal formula iff ϕ is a modal formula

5. $\phi \wedge \psi$ is a modal formula iff ϕ and ψ are modal formulas
6. $\phi \vee \psi$ is a modal formula iff ϕ and ψ are modal formulas
7. $\phi \rightarrow \psi$ is a modal formula iff ϕ and ψ are modal formulas
8. $\phi \leftrightarrow \psi$ is a modal formula iff ϕ and ψ are modal formulas
9. $\Box\phi$ is a modal formula iff ϕ is a modal formula
10. $\Diamond\phi$ is a modal formula iff ϕ is a modal formula ⊣

We use the notation $\phi, \psi, \phi_i, \ldots$ for modal formulas. We often simply abbreviate the name modal formulas to *formulas*.

By a *schema* we mean a set of formulas of a particular form. An *instance* of a schema is thus a member of the set that constitutes the schema. We often identify a schema with the set of its instances, and we make a distinction between schemas and formulas.

The formula $\top$ is called *verum* and $\bot$ *falsum*. $\neg\phi$ is a *negation* of ϕ. $\phi \wedge \psi$ is a *conjunction* of formulas ϕ and ψ and $\phi \vee \psi$ their *disjunction*, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ are their *implication* and *equivalence*, respectively.

Modal operator $\Box$ is essential in modal logic and makes the difference between propositional logic and modal logic.

Formal modal logic represents modalities by means of the unary modal operator, $\Box$. Formula $\Box\phi$ expresses a mode of truth of formula ϕ. This gives the name to the whole discipline – modal logic. Historically, three readings of formula $\Box\phi$ have been very important [2].

First, $\Box$ can be read as "it is necessarily the case that ϕ". Under this reading $\Diamond\phi$ as $\neg\Box\neg\phi$ means "it is not true, that it is necessarily the case that not ϕ", which in fact is "it is possibly the case that ϕ". Now, we would probably regard as correct principle also schema meaning "whatever is necessary is possible". Thus also all instances of schema $\Box\phi \rightarrow \Diamond\phi$ will be contained in our logic. But will we regard generally correct also formulas in form $\Diamond\phi \rightarrow \Box\Diamond\phi$ ("whatever is possible, is necessarily possible") and $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ ("if disjunction of formulas is necessary, then at least one of the disjuncts is necessary")? The status of more complicated schemas is harder to decide with only intuitive understanding of meaning. The precise semantics definition will give us the framework to answer such questions.

Second, in epistemic logic the modal language is used to reason about knowledge. $\Box\phi$ means "the agent knows that ϕ", and we usually write $K\phi$. Now it seems natural to view all instances of $K\phi \rightarrow \phi$ as true, as we are talking about knowledge and not belief and if agent really knows that ϕ, it must hold. On the other hand, an agent is not omniscient, so we would regard $\phi \rightarrow K\phi$ as false. To other and more complicated formulas formal semantics brings clarity too.

Third, in provability logic, $\Box\psi$ is read as "it is provable (in some arithmetical theory) that ψ".

In application fields there are many logics differing in the intuitive meaning of $\Box$, and then the task is to look for set of formulas consisting of just those modal formulas we regard intuitively generally true. We can construct such sets by means of both a syntactical and a semantical framework. In the next section we will define such sets – modal logics.

2.2 Modal logics

A modal logic is a set of formulas in modal language (i.e a subset of Φ^{PL}), which contains at least all propositional tautologies. Sometimes modal logics are called *systems of modal logic*.

Towards a definition of modal logics, first we need the notion of substitution of a formula for all occurrences of propositional letters in another formula.

Definition 2.2.1. *Substitution* is a function $\sigma : \Phi \to \Phi^{ML}$. A substitution σ induces a map $(.)^\sigma : \Phi^{ML} \to \Phi^{ML}$, which is defined recursively:

1. $\bot^\sigma = \bot$
2. $p^\sigma = \sigma(p)$
3. $(\neg\phi)^\sigma = \neg\phi^\sigma$
4. $(\phi \vee \psi)^\sigma = \phi^\sigma \vee \psi^\sigma$
5. $(\Box\phi)^\sigma = \Box\phi^\sigma$

Carrying out a *uniform substitution* σ on formula ϕ is precisely what is defined by $(\phi)^\sigma$. Formula ψ is a *substitution instance* of formula ϕ, if there is some substitution σ such that $\phi^\sigma = \psi$. ⊣

Now, modal logics can be defined formally.

Definition 2.2.2. A set of modal formulas Σ is *modal logic* iff it contains all propositional tautologies and is closed under modus ponens (MP; that is, if $\phi \in \Sigma$ and $\phi \to \psi \in \Sigma$, then $\psi \in \Sigma$) and uniform substitution (that is, if ϕ belongs to Σ, then so do all substitution instances). ⊣

Note that every modal logic contains all substitution instances of the propositional tautologies: for example $\Box p \vee \neg\Box p$ belongs to every modal logic. Even though such substitution instances may contain occurrences of $\Box$ and $\Diamond$, we still call them tautologies. Precise definition follows.

Definition 2.2.3. We call a formula *quasi-atomic* iff it is a propositional variable or formula of the form $\Box\phi$. We will denote the set of all quasi-atomic formulas in Φ^{ML} as Φ^*. $\dashv$

Definition 2.2.4. A *basic assignment* is a function $f^* : \Phi^* \to \{0, 1\}$. A *total assignment* is a function $f : \Phi^{ML} \to \{0, 1\}$ satisfying:

1. there is a basic assignment f^*, such that f and f^* agree on all quasi-atomic formulas, i.e. $f(p^*) = f^*(p^*)$ for every $p^* \in \Phi^*$
2. $f(\bot) = 0$
3. $f(\neg\phi) = \begin{cases} 1 & \text{if } f(\phi) = 0 \\ 0 & \text{otherwise} \end{cases}$
4. $f(\phi \vee \psi) = \begin{cases} 0 & \text{if both} f(\phi) = 0 \text{ and } f(\psi) = 0 \\ 1 & \text{otherwise} \end{cases}$ $\dashv$

Definition 2.2.5. A formula ϕ is a *tautology* iff $f(\phi) = 1$ for all total assignments f. $\dashv$

Within the same framework of total assignments we can define the notion of tautological consequence too.

Definition 2.2.6. A formula ψ is a *tautological consequence* of formulas ϕ_1, ϕ_2, ..., ϕ_n iff for every total assignment function f holds:

$$\text{if } f(\phi_1) = 1, f(\phi_2) = 1, \ldots \text{ and } f(\phi_n) = 1, \text{ then } f(\psi) = 1.$$ $\dashv$

Chellas [1] chose another definition of modal logics, that is equivalent to the one above from [2].

A *rule of inference* has the form

$$\frac{\phi_1, \phi_2, \ldots \phi_n}{\psi}$$

where $n \geq 0$. The formulas $\phi_1, \phi_2, \ldots \phi_n$ are the *hypotheses* of the rule, ψ is the *conclusion.* A set of formulas is said to be *closed under*–, or simply *have*–, a rule of inference just in case it contains the conclusion whenever it contains the hypotheses (or just contain the conclusion if there are no hypotheses – $n = 0$).

Modal logics are equivalently defined in terms of closure under the following rule of inference.

$$\text{RPL.} \quad \frac{\phi_1, \phi_2, \ldots \phi_n}{\psi} \; (n \geq 0)$$

where ψ is a tautological consequence of $\phi_1, \phi_2, \ldots \phi_n$.

Definition 2.2.7. a set of formulas Σ is a *modal logic* iff it is closed under RPL. $\dashv$

Since these definitions are equivalent, we will use any of them, based on our needs.

We denote the set of all tautologies as *propositional logic* and we use an abbreviation PL. Note that we use the same name propositional logic for both discipline – formal system operating with a simple language and the set of all tautologies, as we do with modal logic – a discipline and a set of formulas.

Theorem 2.2.1.

1. *PL is a modal logic. Moreover, it is the smallest modal logic.*
2. *Φ^{PL} is a modal logic, the* inconsistent logic. *This is the biggest modal logic.*
3. *If $\{\Sigma_i \mid i \in I\}$ is a collection*[1] *of modal logics, then $\bigcap_{i\in I} \Sigma_i$ is a modal logic.*

Proof. If A is a tautological consequence of tautologies $A_1, \ldots, A_n$, then A is a tautology too. Thus, PL is closed under RPL and hence is a modal logic. This proves the first claim. Φ^{PL} contains all modal formulas, so it trivially satisfies the definition of modal logic. That the intersection of modal logics is a modal logic, can be easily proved by contradiction. $\square$

(2) and (3) guarantee, that there is a smallest modal logic containing set Γ of formulas. We call this modal logic the *modal logic generated by* Γ. Modal logic generated by an empty set is precisely PL.

Definition 2.2.8. The *theorems* of a modal logic are the formulas in it. We usually write $\vdash_\Sigma \phi$ to denote that ϕ is a theorem of Σ. That means, $\vdash_\Sigma \phi$ iff $\phi \in \Sigma$. $\dashv$

Definition 2.2.9. If Σ_1 and Σ_2 are modal logics and $\Sigma_1 \subseteq \Sigma_2$, we say that Σ_2 is an *extension* of modal logic Σ_1 or that Σ_2 *extends* Σ_1. $\dashv$

2.2.1 Normal modal logics

Normal modal logics are an important class of modal logics. We characterize normal modal logics in terms of the schemas

$$\text{Df}\Diamond. \quad \Box\phi \leftrightarrow \neg\Box\neg\phi$$

and

$$\text{K}. \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

and the rule of inference

$$\text{RN}. \quad \frac{\phi}{\Box\phi}$$

Definition 2.2.10. A modal logic is called *normal* iff it contains the schemas Df$\Diamond$ and K and is closed under RN. $\dashv$

[1] We freely interchange the terms collection, class and set

Theorem 2.2.2.

1. *PL is not a normal logic*

2. *The inconsistent logic is a normal modal logic. This is the biggest modal logic.*

3. *If $\{\Sigma_i \mid i \in I\}$ is a collection of normal modal logics, then $\bigcap_{i\in I} \Sigma_i$ is a normal modal logic.*

Proof. For (1), finding a total assignment f with $f(K) = 0$ we show that K is not a tautology and thus PL is not normal. Such f suffices to satisfy $f(\Box(p \to q)) = 1$, $f(\Box p) = 1$ and $f(\Box q) = 0$. Then $f(\Box(p \to q) \to (\Box p \to \Box q)) = 0$.

(2) is obvious, since the inconsistent logic contains all modal formulas. (3) is reached by an easy proof by contradiction. □

Again, (2) and (3) guarantee, that there is a smallest normal modal logic containing set Γ of formulas. We call this modal logic the *normal modal logic generated by* Γ. Modal logic generated by an empty set is the smallest normal modal logic and it is called K. For every modal logic Σ, $K \subseteq \Sigma$.

To simplify the naming of normal systems we write

$$KS_1 \ldots S_n$$

to denote the normal modal logic obtained by taking the schemas $S_1, \ldots, S_n$ as theorems. In other words, it is the smallest normal modal logic containing (every instance of) the schemas $S_1, \ldots, S_n$.

Stating which formulas generate a logic – by extending the minimal normal logic K with certain schemas of interest – is a usual way of syntactically specifying normal modal logics.

Here are some well-known schemas together with their traditional names.

- D. $\Box\phi \to \Diamond\phi$
- T. $\Box\phi \to \phi$
- B. $\phi \to \Box\Diamond\phi$
- 4. $\Box\phi \to \Box\Box\phi$
- 5. $\Diamond\phi \to \Box\Diamond\phi$

For example, KT4 is the smallest normal logic produced by treating the schemas T and 4 as theorems. (It is also denoted by K4T, the order of the schema names is irrelevant).

Normal modal logics form an important class of modal logics because of their semantic characterization by means of standard frames. We will introduce the standard frames in section 2.3.1. Then we will consider the notion of soundness and completeness in 2.4.

2.2.2 Non-normal modal logics

Since every normal modal logic is a modal logic and PL is not normal we know that normal modal logics form a proper subclass of all modal logics. Under non-normal modal logics one can understand the class of modal logics that aren't normal. Although it may be a little confusing first, under non-normal modal logics we consider classical modal logics.

We define classical modal logics in terms of the schema Df$\Diamond$ and the rule of inference

$$\text{RE.} \quad \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

Definition 2.2.11. A modal logic is *non-normal* iff it contains Df$\Diamond$ and is closed under RE. $\dashv$

Classical modal logics form a superset of the set of all normal modal logics, but a subset of the set of all modal logics. That both these inclusions are in fact proper, we will show in section 2.4.3.

Theorem 2.2.3. *Every normal modal logic* Σ *is classical.*

Proof. Both normal and classical modal logics contain Df$\Diamond$. Now we show that whenever a modal logic is closed under RN, it is closed under RE too. PL in the annotation of the proof below means we make use of propositional logic.

1.	$\phi \leftrightarrow \psi$	hypothesis
2.	$\phi \rightarrow \psi$	1, PL
3.	$\Box(\phi \rightarrow \psi)$	2, RN
4.	$\Box\phi \rightarrow \Box\psi$	3, K, PL
5.	$\psi \rightarrow \phi$	1, PL
6.	$\Box(\psi \rightarrow \phi)$	5, RN
7.	$\Box\psi \rightarrow \Box\phi$	6, K, PL
8.	$\Box\psi \leftrightarrow \Box\phi$	4, 7, PL

$\Box$

Definition 2.2.12. We say that logic Σ_1 is *weaker* than Σ_2 iff $\Sigma_1 \subseteq \Sigma_2$. If the inclusion is proper, it is *strictly weaker.* $\dashv$

Non-normal modal logics as such are logics that aren't normal, i.e. systems strictly weaker than the smallest normal logic K. Some of non-normal modal logics are classical, the others are not.

The existence of semantics (section 2.3.2) of classical modal logics provides another perspective on investigating these logics. For non-classical logics, this perspective is missing.

Normal modal logics are a special case of classical modal logics. That means the tools for analyzing classical logics can be used for normal systems as well. However, for analyzing

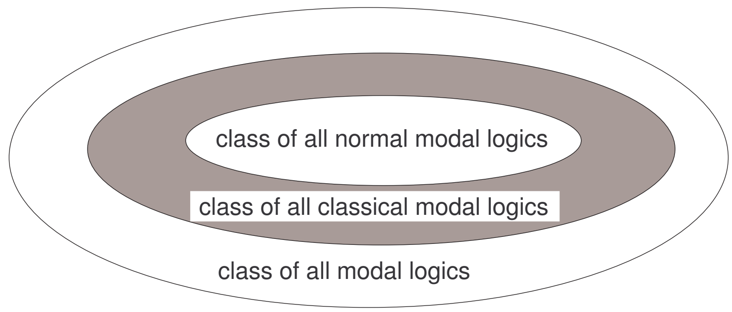

Figure 2.1: Gradual inclusion of classes of all, classical and normal modal logics

normal logics we have stronger tools and techniques (see a broad repertoire of model-theory techniques in [2]).

Therefore from now on, we will identify non-normal modal logics with the class of all classical modal logics.

Now we will prove a theorem that will guarantee the existence of the smallest classical logic.

Theorem 2.2.4. *If $\{\Sigma_i \mid i \in I\}$ is a collection of classical modal logics, then $\bigcap_{i\in I} \Sigma_i$ is a classical modal logic.*

Proof. Again, simple proof by contradiction suffices. □

As a corollary of the theorem 2.2.4 we get the existence of the smallest classical modal logic. We denote this logic E.

As with normal modal logics we simplify the naming of classical systems. We write

$$ES_1 \ldots S_n$$

to denote the classical modal logic obtained by taking the schemas $S_1, \ldots, S_n$ as theorems.

From the picture 2.2.2 we can see the relation between classes of all, classical and normal modal logics. Every normal logic is classical and every classical modal logic is a modal logic.

Let's take a look on individual modal logics. We know that the more axioms we a logic has the more theorems it contains. The smallest modal logic is PL, and it is contained (as a subset) in every modal logic. E is the smallest classical modal logic. As E is included in every classical modal logic, it is included in every normal modal logic. In particular, E is included in K. We have shown, that:

$$PL \subseteq E \subseteq K$$

Proper inclusion of the respective logics we can show with help of the soundness and completeness theorems in 2.4.3. In fact PL is strictly weaker than E and that is strictly weaker than K.

Classical modal logics represent an interesting class of modal logic because of their semantic characterization. They are related to the neighbourhood frames. We will introduce the neighbourhood frames in subsection 2.3.2. Then we will consider the notion of soundness and completeness in 2.4.

2.3 Semantics - frames, models, truth, validity

First we introduce the standard frame semantics – semantics of normal modal logics. In the second part of this section we define neighbourhood semantics – semantical characterization of classical modal logics.

2.3.1 Standard frames

The standard semantics was proposed by American philosopher Saul Kripke in the late 1950's and early 1960's. It was originally developed for modal logics, but it was later adapted to intuitionist logic and some other formal systems. It is also known as *Kripke semantics* or *relational semantics*. We will use the terms *standard frame semantics* or where it can cause no confusion we may drop the prefix standard and refer to it as *frame semantics* (frames, models, etc.).

A standard frame is a relational structure, i.e. a set with a binary relation on it.

Definition 2.3.1. A *standard frame* is a pair $\mathcal{F} = (W, R)$ such that

1. W is a non-empty set.
2. R is a binary relation on W, i.e. $R \subseteq W \times W$ $\dashv$

We call elements of W *worlds* or *states* and use the notation $w, v, v_i \dots$. Relation R is called *accessibility relation*. If $(w, v) \in R$ we say that state v is *accessible* or *reachable* from state w and denote this by Rwv. We use notation $\mathcal{F}$ for standard frames.

A standard model is a standard frame with a *valuation*.

Definition 2.3.2. A *standard model* is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F} = (W, R)$ is a frame and V is a function assigning to each propositional variable p a subset $V(p)$ of W. $\dashv$

We use notation $\mathcal{M}$ for standard models. Valuation V specifies for each propositional variable p the set of worlds, in which p is true.

We can specify a model based on frame $\mathcal{F} = (W, R)$ either by couple $\mathcal{M} = (\mathcal{F}, V)$ or, equivalently, by triple $\mathcal{M} = (W, R, V)$. We use both types of notation.

The notion of truth is defined via the satisfaction relation, which relates models, worlds and formulas as follows.

Definition 2.3.3. Suppose w is a world in a model $\mathcal{M} = (W, R, V)$. Then we inductively define the notion of a formula ϕ being *satisfied* (or *true*) in $\mathcal{M}$ in world w as follows:

1. $\Vdash_w^{\mathcal{M}} p$ iff $w \in V(p)$, where p is a propositional variable
2. $\Vdash_w^{\mathcal{M}} \bot$ never
3. $\Vdash_w^{\mathcal{M}} \neg\phi$ iff not $\Vdash_w^{\mathcal{M}} \phi$
4. $\Vdash_w^{\mathcal{M}} \phi \vee \psi$ iff $\Vdash_w^{\mathcal{M}} \phi$ or $\Vdash_w^{\mathcal{M}} \psi$
5. $\Vdash_w^{\mathcal{M}} \Box\phi$ iff for all $v \in W$ with Rwv we have $\Vdash_v^{\mathcal{M}} \phi$
6. $\Vdash_w^{\mathcal{M}} \Diamond\phi$ iff for some $v \in W$ with Rwv we have $\Vdash_v^{\mathcal{M}} \phi$ ⊣

Note that by definition of $\Vdash_w^{\mathcal{M}} \Diamond\phi$ reflects the duality of $\Diamond$ and $\Box$.

Example 2.3.1. Consider standard model $\mathcal{M} = (W, R, V)$, where $W = \{w_1, w_2, w_3\}$, $R = \{(w_1, w2), (w_1, w_3)\}$ and $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_1, w_3\}$.

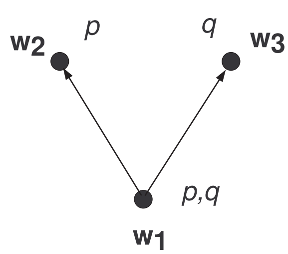

Figure 2.2: An example of standard model

In this model

- $\Vdash_{w_1}^{\mathcal{M}} p \wedge q$, because $\Vdash_{w_1}^{\mathcal{M}} p$ and $\Vdash_{w_1}^{\mathcal{M}} q$
- $\nVdash_{w_2}^{\mathcal{M}} p \wedge q$, because $\nVdash_{w_2}^{\mathcal{M}} q$
- $\Vdash_{w_1}^{\mathcal{M}} \Box(p \vee q)$, because both $\Vdash_{w_2}^{\mathcal{M}} p \vee q$ and $\Vdash_{w_3}^{\mathcal{M}} p \vee q$
- $\Vdash_{w_2}^{\mathcal{M}} \Box(p \vee q)$ because there are no states accessible from w_2
- $\Vdash_{w_1}^{\mathcal{M}} \Diamond(p \vee q)$, because there is a state w_2 such that Rw_1w_2 and $\Vdash_{w_2}^{\mathcal{M}} p \vee q$
- $\nVdash_{w_2}^{\mathcal{M}} \Diamond(p \vee q)$, because there is no such state v that Rw_1v and $\Vdash_v^{\mathcal{M}} p \vee q$ ⊣

The notion of validity is an analogue to tautologies in propositional logic. Just like a propositional tautology which is true under all valuations, formula valid in a frame is true with no regard to the valuation and world.

Definition 2.3.4. A formula ϕ is *valid in a world w in a frame $\mathcal{F}$* (notation $\Vdash^{\mathcal{F}}_w \phi$) if ϕ is true at w in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$; ϕ is *valid in a frame $\mathcal{F}$* (notation $\Vdash^{\mathcal{F}} \phi$) if it is valid in every world in $\mathcal{F}$. A formula ϕ is *valid on a class of frames C* (notation $\Vdash^{C} \phi$) if it is valid in every frame $\mathcal{F}$ in C. $\dashv$

All tautologies are certainly valid in every standard frame. But there are also other formulas, of purely modal character, that are valid in every frame.

One of such formulas is schema K: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. To see this, take any frame $\mathcal{F}$, any state w and any valuation V. We have to show, that if $\Vdash^{(\mathcal{F},V)}_w \Box(\phi \to \psi)$, then $\Vdash^{(\mathcal{F},V)}_w \Box\phi \to \Box\psi$. So assume by contradiction that there are formulas ϕ, ψ such that $\Vdash^{(\mathcal{F},V)}_w \Box(\phi \to \psi)$ and $\nVdash^{(\mathcal{F},V)}_w \Box\phi \to \Box\psi)$. That means $\Vdash^{(\mathcal{F},V)}_w \Box\phi$ and $\nVdash^{(\mathcal{F},V)}_w \Box\psi$. Then, by definition there is a state v such that Rwv and have $\nVdash^{(\mathcal{F},V)}_v \psi$. And since $\Vdash^{(\mathcal{F},V)}_v \phi \to \psi$ because it holds in all states accessible from w, in v the formula ϕ must be false. But that's a contradiction to $\nVdash^{(\mathcal{F},V)}_w \Box\phi$.

The schema $\Box\phi \to \Box\Box\phi$ (known as 4) is not valid in all frames. We present a frame $\mathcal{F}$, a state w and a valuation V and an instance of the schema that is not satisfied in state w. Let $\mathcal{F}$ be a three-state frame with universe $\{w_1, w_2, w_3\}$, and relation $\{(w_1, w_2), (w_2, w_3)\}$. Let V be any valuation on $\mathcal{F}$ such that $V(p) = \{w_2\}$. Then $\Vdash^{(\mathcal{F},V)}_{w_1} \Box p$ but $\nVdash^{(\mathcal{F},V)}_{w_1} \Box\Box p$.

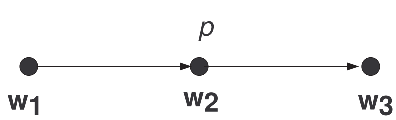

Figure 2.3: Model falsifying validity of 4 in the class of all frames

But there is a class of frames on which $\Box\phi \to \Box\Box\phi$ is valid – the class of transitive frames. We call a frame $\mathcal{F} = (W, R)$ transitive, iff its accessibility relation R is transitive. We will show that $\Box\phi \to \Box\Box\phi$ is valid in every transitive frame. Suppose by contradiction, it is not. Then there is a frame $\mathcal{F}$, valuation V and world w such that $\Vdash^{(\mathcal{F},V)}_w \Box\phi$ but $\nVdash^{(\mathcal{F},V)}_w \Box\Box\phi$. From the latter there is state v, Rwv such that $\nVdash^{(\mathcal{F},V)}_v \Box\phi$ that means there is a state u, Rvu such that $\nVdash^{(\mathcal{F},V)}_u \phi$. But since R is transitive, also Rwu and therefore $\nVdash^{(\mathcal{F},V)}_w \Box\phi$ which is a contradiction.

Theorem 2.3.1. *The following schemas (defined in section 2.2.1) are valid respectively in the indicated classes of standard frames*

K	–	*all*	–	–
D	–	*serial*	–	$\forall u \exists v (Ruv)$
T	–	*reflexive*	–	$\forall u (Ruu)$
B	–	*symmetric*	–	$\forall uv (Ruv \to Rvu)$
4	–	*transitive*	–	$\forall uvw ((Ruv \wedge Rvw) \to Ruw)$
5	–	*euclidean*	–	$\forall uvw ((Ruv \wedge Ruw) \to Rvw)$

Proof. To be found in [1], theorem 3.5. □

2.3.2 Neighbourhood frames

Neighbourhood semantics is generalization of standard frame semantics. It was originally developed by Dana Scott and Richard Montague in 1970 and it is also known as Scott-Montague semantics. Chellas [1] calls neighbourhood models minimal models. In our work we use the name *neighbourhood semantics*.

Motivation for establishing this semantics was semantic characterization of modal logics weaker than K, the smallest normal modal logic. Indeed, neighbourhood frames provide semantical characterization for classical modal logics (see 2.2.2).

Like in standard frames a neighbourhood frame has a set of worlds, but instead of an accessibility relation it has a neighbourhood function $N : W \to \mathscr{P}(\mathscr{P}(W))$.

Definition 2.3.5. A *neighbourhood frame* is a pair $\mathfrak{F} = (W, N)$ such that

1. W is a non-empty set.
2. N is a mapping from W to sets of subsets of W (i.e. $N(w) \subseteq \mathscr{P}(W)$, for each world w in W). ⊣

Intuitively, function N assigns to a world w a collection of subsets of W. Every subset represents a proposition in some sense necessary for world w. As we will shortly see in the definition of satisfaction, the proposition is represented by set of world, where a formula is true.

Example 2.3.2. An example of a neighbourhood frame is $\mathfrak{F} = (W, N)$ on three-world set $W = \{w_1, w_2, w_3\}$. Neighbourhoods of the worlds are as follows:

1. $N(w_1) = \emptyset$
2. $N(w_2) = \{\emptyset, \{w_1\}\}$
3. $N(w_3) = \{\{w_1\}, \{w_1, w_3\}\}$. ⊣

As with standard semantics, a neighbourhood model is a neighbourhood frame with a valuation.

Definition 2.3.6. A *neighbourhood model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F} = (W, N)$ is a neighbourhood frame and V is a function assigning to each propositional variable p a subset $V(p)$ of W. ⊣

Definition of satisfaction of a formula in a state in a neighbourhood model follows.

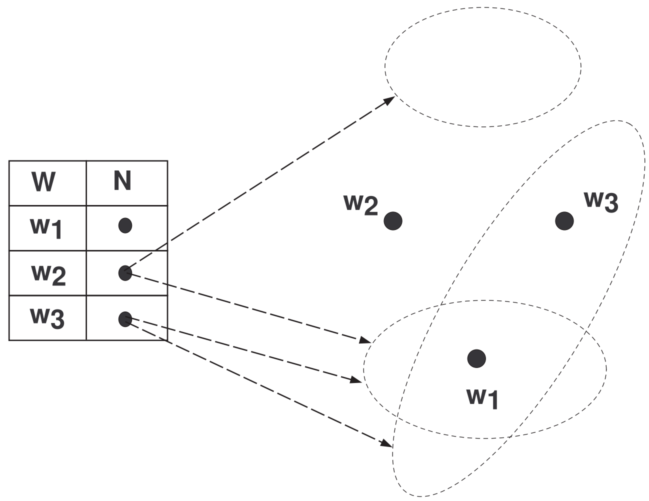

Figure 2.4: An example of neighbourhood frame

Definition 2.3.7. Suppose w is a world in a neighbourhood model $\mathfrak{M} = (W, N, V)$. Let $\|\phi\|^{\mathfrak{M}}$ be *truth set of formula ϕ in the model $\mathfrak{M}$* and stand for $\{w \in \mathfrak{M} \mid \models_w^{\mathfrak{M}} \phi\}$.

Then we inductively define the notion of a formula ϕ being *satisfied* (or *true*) in $\mathfrak{M}$ at world w as follows:

1. $\models_w^{\mathfrak{M}} p$ iff $w \in V(p)$, where p is a propositional variable
2. $\models_w^{\mathfrak{M}} \bot$ never
3. $\models_w^{\mathfrak{M}} \neg\phi$ iff not $\models_w^{\mathfrak{M}} \phi$
4. $\models_w^{\mathfrak{M}} \phi \vee \psi$ iff $\models_w^{\mathfrak{M}} \phi$ or $\models_w^{\mathfrak{M}} \psi$
5. $\models_w^{\mathfrak{M}} \Box\phi$ iff $\|\phi\|^{\mathfrak{M}} \in N(w)$
6. $\models_w^{\mathfrak{M}} \Diamond\phi$ iff $(W - \|\phi\|^{\mathfrak{M}}) \notin N(w)$ ⊣

The truth set of a formula is a set of worlds where the formula is true. From the definition of satisfaction of formula $\Box\phi$ we can see that it is true in a world w if the proposition expressed by ϕ – the truth set of ϕ – is amongst those necessary, specified in the neighbouurhood $N(w)$.

The definition of formula $\Diamond\phi$ being satisfied in world w reflects the duality of $\Box$ and $\Diamond$. That means $\Diamond$ should be true in a world whenever $\neg\Box\neg\phi$ is. Really, $\neg\Box\neg\phi$ is true in w iff it is not the case that $\|\neg\phi\|^{\mathcal{M}}$ belongs to $N(w)$. Thus it is not the case that $W - \|\phi\|^{\mathcal{M}}$ (see theorem below) belongs to $N(w)$ which is precisely the definition.

Example 2.3.3. Consider a neighbourhood model $\mathfrak{M} = (W, N, V)$, where: $W = \{w_1, w_2, w_3\}$, $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_1, w_3\}$ and neighbourhoods of the worlds are as follows $N(w_1) = \emptyset$, $N(w_2) = \{\emptyset, \{w_1\}\}$ and $N(w_3) = \{\{w_1\}, \{w_1, w_3\}\}$.

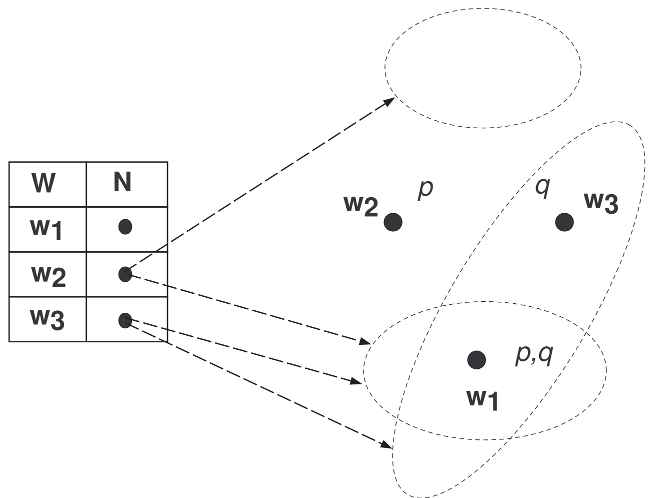

Figure 2.5: An example of neighbourhood model

In this neighbourhood model holds:

- $\vDash^{\mathfrak{M}}_{w_1} p \wedge q$, because $\vDash^{\mathfrak{M}}_{w_1} p$ and $\vDash^{\mathfrak{M}}_{w_1} q$
- $\nvDash^{\mathfrak{M}}_{w_2} p \wedge q$, because $\nvDash^{\mathcal{M}}_{w_2} q$
- $\vDash^{\mathfrak{M}}_{w_3} \Box(p \wedge q)$, because $\|p \wedge q\|^{\mathfrak{M}} = \|p\|^{\mathfrak{M}} \cap \|q\|^{\mathfrak{M}} = \{w_1, w_2\} \cap \{w_1, w_3\} = \{w_1\}$ belongs to $N(w_3)$
- $\vDash^{\mathfrak{M}}_{w_3} \Diamond p$ because $W - \|p\|^{\mathfrak{M}} = W - \{w_1, w_2\} = \{w_3\} \notin N(w_3)$ $\dashv$

The validity is defined in the same way it was defined in frame semantics.

Definition 2.3.8. A formula ϕ is *valid in a world w in a neighbourhood frame* $\mathfrak{F}$ (notation $\vDash^{\mathfrak{F}}_{w} \phi$) iff ϕ is true at w in every neigbourhood model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$; ϕ is *valid in a neihgbourhood frame* $\mathfrak{F}$ (notation $\vDash^{\mathfrak{F}} \phi$) if it is valid at every state in $\mathfrak{F}$. A formula ϕ is *valid on a class of neighbourhood frames* C (notation $\vDash^{C} \phi$) if it is valid in every neighbourhood frame $\mathfrak{F}$ in C. $\dashv$

As in standard frame semantics, we can construct class of frames by specifying condition on the frame. In this case, we put a condition on the neighbourhoods of worlds. Conditions for the important schemas to be validated in classes of neighbourhood frames are described by following theorem.

Theorem 2.3.2. *The following schemas (defined in section 2.2.1) are valid respectively in the indicated classes of neighbourhood frames. (conditions hold for every world w and proposition X in neighbourhood frame (W,N); $-X$ means $W - X$)*

D	–	*if* $X \in N(w)$, *then* $-X \notin N(w)$
T	–	*if* $X \in N(w)$, *then* $w \in X$
B	–	*if* $w \in X$, *then* $\{v \in W \mid -X \notin N(v)\} \in N(w)$
4	–	*if* $X \in N(w)$, *then* $\{v \in W \mid X \in N(v)\} \in N(w)$
5	–	*if* $X \notin N(w)$, *then* $\{v \in W \mid X \notin N(v)\} \in N(w)$

Proof. To be found in [1], theorem 7.11. □

Now we will prove a theorem that reveals the structure of truth sets of some sorts of formulas.

Theorem 2.3.3. *Let $\mathfrak{M}$ be a neighbourhood model. Then*

1. $\|p\|^{\mathfrak{M}} = V(p)$
2. $\|\top\|^{\mathfrak{M}} = W$
3. $\|\bot\|^{\mathfrak{M}} = \emptyset$
4. $\|\neg\phi\|^{\mathfrak{M}} = W - \|\phi\|^{\mathfrak{M}}$
5. $\|\phi \wedge \psi\|^{\mathfrak{M}} = \|\phi\|^{\mathfrak{M}} \cap \|\psi\|^{\mathfrak{M}}$
6. $\|\phi \vee \psi\|^{\mathfrak{M}} = \|\phi\|^{\mathfrak{M}} \cup \|\psi\|^{\mathfrak{M}}$
7. $\|\phi \to \psi\|^{\mathfrak{M}} = (W - \|\phi\|^{\mathfrak{M}}) \cup \|\psi\|^{\mathfrak{M}}$
8. $\|\phi \leftrightarrow \psi\|^{\mathfrak{M}} = ((W - \|\phi\|^{\mathfrak{M}}) \cup \|\psi\|^{\mathfrak{M}}) \cap ((W - \|\psi\|^{\mathfrak{M}}) \cup \|\phi\|^{\mathfrak{M}})$

Proof.

1. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} p\}$ is exactly $V(p)$
2. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \top\} = W$, because $\models_w^{\mathfrak{M}} \top$ is true at every world.
3. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \bot\} = \emptyset$, because $\models_w^{\mathfrak{M}} \bot$ is true at no world.
4. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \neg\phi\} = W - \{w \in \mathfrak{M} : \Vdash_w^{\mathfrak{M}} \phi\}$, as $\models_w^{\mathfrak{M}} \neg\phi$ is by definition true iff $\not\models_w^{\mathfrak{M}} \phi$.
5. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \phi \wedge \psi\} = \{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \phi\} \cap \{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \psi\}$, as $\models_w^{\mathfrak{M}} \phi \wedge \psi$ is by definition true iff $\models_w^{\mathfrak{M}} \phi$ and $\models_w^{\mathfrak{M}} \psi$.
6. $\{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \phi \vee \psi\} = \{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \phi\} \cup \{w \in \mathfrak{M} : \models_w^{\mathfrak{M}} \psi\}$, as $\models_w^{\mathfrak{M}} \phi \vee \psi$ is by definition true iff $\models_w^{\mathfrak{M}} \phi$ or $\models_w^{\mathfrak{M}} \psi$.

7. $\{w \in \mathfrak{M} : \vDash_w^{\mathfrak{M}} \phi \to \psi\} = (W - \|\phi\|^{\mathfrak{M}}) \cup \|\psi\|^{\mathfrak{M}}$ because $(\phi \to \psi)$ iff $(\neg\phi \vee \psi)$.
8. $\{w \in \mathfrak{M} : \vDash_w^{\mathfrak{M}} \phi \to \psi\} = ((W - \|\phi\|^{\mathfrak{M}}) \cup \|\psi\|^{\mathfrak{M}}) \cap ((W - \|\psi\|^{\mathfrak{M}}) \cup \|\phi\|^{\mathfrak{M}})$ because $(\phi \leftrightarrow \psi)$ iff $((\phi \to \psi) \wedge (\psi \to \phi))$.

□

2.4 Soundness and completeness

As we saw in section 2.2, modal logics are sets of formulas satisfying certain simple closure conditions. They can be specified either syntactically or semantically, and this gives rise to natural questions:

- Given a semantically specified logic, can we give it a syntactic characterization?
- Given a syntactically specified logic, can we give it a semantic characterization (in terms of standard or neighbourhood frames)?

To answer this type of questions, we need to prove soundness and completenes theorems about the particular logics.

Now we will define soundness and completeness of normal modal logics.

2.4.1 Soundness

First, we will denote all formulas valid on a class of frames the logic of this class.

Definition 2.4.1. The set Σ_C of all formulas that are valid on a class C of frames is called the *logic of* C. ⊣

Definition 2.4.2. Let C be a class of frames. A normal modal logic Σ is *sound* with respect to C iff $\Sigma \subseteq \Sigma_C$. Equivalently Σ is sound with respect to C iff for all formulas ϕ and all frames $\mathcal{F}$ in C, $\vdash_\Sigma \phi$ implies $\Vdash^{\mathcal{F}} \phi$. If Σ is sound with respect to C, we say that C is a *class of frames for* Σ. ⊣

Example 2.4.1. The following normal modal logics are sound with respect to the respective classes of frames:

K	–	all frames
KD	–	serial frames
KT	–	reflexive frames
KB	–	symmetric frames
$K4$	–	transitive frames

⊣

Theorem 2.4.1. *Modus ponens, generalization and uniform substitution preserve validity on any class of frames.*

Proof. For modus ponens, suppose that $\Vdash^C \phi, \Vdash^C \phi \to \psi$, we want to show that $\Vdash^C \psi$. Take any frame $\mathcal{F}$, any world w any valuation V. We know that $\Vdash_w^{(\mathcal{F},V)} \phi$ and $\Vdash_w^{(\mathcal{F},V)} \phi \to \psi$. Thus, we have $\Vdash_w^{(\mathcal{F},V)} \psi$. Since w, V and $\mathcal{F}$ was arbitrary, we proved that $\Vdash^C \psi$.

For generalization, suppose that $\Vdash^C \phi$, we want to show that $\Vdash^C \Box\phi$. Take any frame $\mathcal{F}$, any world w any valuation V. We know that $\Vdash_w^{(\mathcal{F},V)} \phi$. Since $\Vdash^{\mathcal{F}} \phi$ we know that for every world v, $\Vdash_v^{(\mathcal{F},V)} \phi$. That means also for particular worlds u, such that Rwu, phi is true in u. Hence, $\Vdash_w^{(\mathcal{F},V)} \Box\phi$.

Finally, for uniform substitution, suppose that $\Vdash^C \phi$, we want to show that $\Vdash^C \psi$, where ψ is a substitution instance of ϕ. Take any frame $\mathcal{F}$. Since ϕ is true in any world under any valuation of its propositional variables, it will remain true after uniform substitution of any formulas for these propositional variables. Thus, $\Vdash_w^{(\mathcal{F},V)} \psi$. □

Soundness is generally easily demonstrated. We have to show that the axioms are valid, and that the three inference rules (MP, generalization, and uniform substitution) preserve validity on the class of frames in question. Previous theorem says, that these inference rules preserve validity on any class of frames, so proving validity reduces to checking the validity of axioms. We checked validity of schemas in the previous example in 2.3.1.

We have just proved the following theorem.

Theorem 2.4.2. *Let $S_1, \ldots, S_n$ be schemas valid respectively in classes of standard frames $C_1, \ldots, C_n$. Then the modal logic $KS_1 \ldots S_n$ is sound with respect to the class $C_1 \cap \ldots \cap C_n$.*

2.4.2 Completeness

We begin with definition of semantic consequence, also called semantic entailment. Then we define deducibility and consistence.

Definition 2.4.3. Let C be a class of frames. Let Δ and ϕ be a set of formulas and a single formula. We say that ϕ is a *local semantic consequence of* Δ *over* C (notation $\Delta \Vdash^S \phi$) if for all models $\mathcal{M}$ from C and all worlds $w \in \mathcal{M}$, if $\Vdash_w^{\mathcal{M}} \Delta$ then $\Vdash_w^{\mathcal{M}} \phi$ ⊣

Definition 2.4.4. A formula ϕ is Σ-*deducible* from a set of formulas Γ – written $\Gamma \vdash_\Sigma \phi$ – iff Σ contains a theorem of the form

$$(\phi_1 \wedge \ldots \wedge \phi_n) \to \phi,$$

where the conjuncts ϕ_i of the antecedent are formulas in Γ. ⊣

Definition 2.4.5. A set of formulas Γ is Σ-*consistent* – written $\mathrm{Con}_\Sigma\Gamma$ – iff formula $\bot$ is not Σ-deducible from Γ. Thus Γ is Σ-*inconsistent* just when $\Gamma \vdash_\Sigma \bot$. ⊣

Now we can define strong and weak completeness.

Definition 2.4.6. Let C be a class of frames. A logic Σ is *strongly complete* with respect to C iff for any set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \Vdash^C \phi$ then $\Gamma \vdash_\Sigma \phi$. That is, if Γ semantically entails ϕ on C, then ϕ is Σ-deducible from Γ.

A logic Σ is *weakly complete* with respect to C iff for any formula ϕ, if $\Vdash^C \phi$ then $\vdash_\Sigma \phi$. $\dashv$

Note that weak completeness is a special case of strong completeness in which Γ is empty, thus strong completeness implies weak completeness. (The converse does not hold.)

Note that the definition of weak completeness can be reformulated to the style of definition of soundness. Σ is weakly complete with respect to C if $\Sigma_S \subseteq \Sigma$. Thus if we prove that a syntactically specified logic Σ is both sound and weakly complete with respect to some class of frames C, we have established a perfect match between the syntactical and semantical perspectives, $\Sigma = \Sigma_S$.

Following theorem is frequently used for the proof of completeness.

Definition 2.4.7. A formula ϕ is *satisfiable* on a frame $\mathcal{F}$ if there is a model $\mathcal{M}$ based on $\mathcal{F}$ and a world w in $\mathcal{M}$ such that $\Vdash^{\mathcal{M}}_w \phi$, i.e. if exists a valuation making ϕ true in w.

Theorem 2.4.3. *A logic Σ is strongly complete with respect to a class of frames C iff every Σ-consistent set of formulas is satisfiable on some $\mathcal{F} \in C$. Σ is weakly complete with respect to a class of frames C iff every Σ-consistent formula is satisfiable on some $\mathcal{F}$ in C.*

Proof. The result for weak completeness follows from the one for strong completeness, so we examine only the latter. To prove the right to left implication we argue by contraposition. Suppose Σ is not strongly complete with respect to C. Thus there is a set of formulas $\Gamma \cup \{\phi\}$ such that $\Gamma \Vdash^C \phi$ but $\Gamma \nvdash_\Sigma \phi$. Then $\Gamma \cup \{\neg\phi\}$ is Σ-consistent, but not satisfiable on any frame in C.

For the left to right direction suppose by contradiction, that there is a Σ-consistent set $\Gamma \cup \{\phi\}$ such that it is not satisfiable on any frame in C. Thus $\Gamma \cup \{\neg\phi\}$ is valid on class C. Formula $\neg\phi$ is valid in class C, thus also $\Gamma \Vdash^C \neg\phi$. Therefore $\Gamma \vdash_\Sigma \neg\phi$, hence $\Gamma \cup \{\phi\}$ is inconsistent. □

The content of theorem 2.4.3 is that completeness theorems are essentially model existence theorems: Given a normal logic Σ, we prove its strong completeness with respect to some class of frames by showing that every Σ-consistent set of formulas can be satisfied in some suitable model. Thus the question is, how to build suitable satisfying models. There is an answer: out of maximal consistent sets of formulas and build canonical models.

Definition 2.4.8. A set of formulas Γ is *maximal Σ-consistent* if Γ is Σ-consistent, and any set of formulas properly containing Γ is Σ-inconsistent. If Γ is a maximal Σ-consistent set of formulas, then we say it is a Σ-MCS. $\dashv$

Intuitively, a set is a MCS if it is consistent and contains as many formulas as it can without becoming inconsistent. Maximal consistent sets have following properties.

Theorem 2.4.4. *If Σ is a logic and Γ is a Σ-MCS, then:*

1. *$\phi \in \Gamma$ iff $\Gamma \vdash_\Sigma \phi$;*

2. *Γ is closed under modus ponens: if ϕ, $\phi \to \psi \in \Gamma$, then $\psi \in \Gamma$;*

3. *$\Sigma \subseteq \Gamma$;*

4. *for all formulas ϕ: $\phi \in \Gamma$ or $\neg\phi \in \Gamma$, but not both;*

5. *for all formulas ϕ, ψ: $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.*

Proof. (1) is a technical exercise, uses properties of consistency and deducibility. (2) comes from (1) and the fact that if $\Gamma \vdash_\Sigma \phi$ and $\Gamma \vdash_\Sigma \phi \to \psi$ then $\Gamma \vdash_\Sigma \psi$.(3) comes from (1) and the fact that $\vdash_\Sigma \phi$ iff for every Γ, $\Gamma \vdash_\Sigma \phi$. For (4), if none of ϕ, $\neg\phi$ were included, Γ wouldn't be maximal. If both were included, Γ would be inconsistent. For (5), using (1) by contradiction we prove both directions. □

The next theorem is known as *Lindenbaum's Lemma*. It states that every consistent set of formulas has a maximal extension.

Theorem 2.4.5 (Lindenbaum's Lemma). *If Δ is a Σ-consistent set of formulas, then there is a Σ-MCS Δ^+ such that $\Delta \subseteq \Delta^+$.*

Proof. Let $\phi_0, \phi_1, \phi_2, \dots$ be an enumeration of the formulas of our language. We define the set Δ^+ as the union of a chain of Σ-consistent sets as follows:

$$\begin{array}{lcl} \Delta_0 & = & \Delta \\ \Delta_{n+1} & = & \begin{cases} \Delta_n \cup \{\phi_n\} & \text{if this is } \Sigma\text{-consistent} \\ \Delta_n \cup \{\neg\phi_n\} & \text{otherwise} \end{cases} \\ \Delta^+ & = & \bigcup_{n>=0} \Delta_n. \end{array}$$

Clearly $\Delta \subseteq \Delta^+$. The following properties imply, that Δ^+ is a Σ-MCS.

1. Δ^+ is Σ-consistent. Assume Δ^+ is Σ-inconsistent. Then a finite subset $\Gamma \subseteq \Delta^+$ is inconsistent. But then $\Gamma \subseteq \Delta_n$ for some $n \geq 0$, which is a contradiction, since all Δ_n are Σ-consistent by definition.

2. Δ^+ is maximal. Let $\phi \notin \Delta^+$ and $\phi = \phi_n$. Since $\phi_n \notin \Delta^+ \supseteq \Delta_{n+1}$, we have that $\Delta_{n+1} \cup \{\phi_n\}$ is inconsistent. Thus, so is $\Delta^+ \cup \{\phi_n\}$.

□

Now we can define canonical models.

Definition 2.4.9. The *canonical model* $\mathcal{M}^\Sigma$ for a normal modal logic Σ is the triple $(W^\Sigma, R^\Sigma, V^\Sigma)$ where

1. W^Σ is the set of all Σ-MCSs;

2. R^Σ is the binary relation on W^Σ defined by $R^\Sigma wu$ if for all formulas ψ, $\psi \in u$ implies $\Diamond\psi \in w$. R^Σ is called the *canonical relation*;

3. V^Σ is the valuation defined by $V^\Sigma(p) = \{w \in W^\Sigma \mid p \in w\}$. V^Σ is called the *canonical valuation*.

The pair $\mathcal{F}^\Sigma = (W^\Sigma, R^\Sigma)$ is called the *canonical frame* for Σ. $\dashv$

The canonical valuation equates the truth of a propositional variable at w with its membership in w. Our ultimate goal is to prove a truth lemma which will lift this "truth=membership" equation to arbitrary formulas.

Second, note that the states of $\mathcal{M}^\Sigma$ consist of all Σ-consistent MCSs. The significance is that by Lindenbaum's Lemma, any Σ-consistent set of formulas is a subset of some point in $\mathcal{M}^\Sigma$ – hence, by truth lemma proved below any Σ-consistent set of formulas is true at some point in this model. In short, the single structure $\mathcal{M}^\Sigma$ is a universal model for the logic Σ, which is why it is called canonical.

Finally, consider the canonical relation: a state w is related to a state u precisely when for each formula $\psi \in u$, w contains the information $\Diamond\psi$. intuitively, this captures what we mean by MCSs being coherently related.

Theorem 2.4.6. *For any normal modal logic Σ, $R^\Sigma wv$ iff for all formulas ψ, $\Box\psi \in w$ implies $\psi \in v$.*

Proof. For the left to right direction, suppose $R^\Sigma wv$. Further suppose $\psi \notin v$. As v is an MCS, by proposition 4.16 $\neg\psi \in v$. As $R^\Sigma wv$, $\Diamond\neg\psi \in w$. As w is consistent, $\neg\Diamond\neg\psi \notin w$. That is, $\Box\psi \notin w$ and we have established the contrapositive. We leave right to left for the reader. $\Box$

In fact the definition of R^Σ is exactly what we require. All that remains to be checked is that enough coherently related MCSs exist for our purposes.

Theorem 2.4.7 (Existence Lemma). *For any normal modal logic Σ and any state w in W^Σ, if $\Diamond\psi \in w$, then there is a state v in W^Σ such that $R^\Sigma wv$ and $\psi \in v$.*

Proof. Suppose $\Diamond\phi \in w$. We will construct a state v such that $R^\Sigma wv$ and $\phi \in v$.

Let v^- be $\{\phi\} \cup \{\psi \mid \Box\psi \in w\}$. Then v^- is consistent. Suppose, it's not. Then there are $\psi_1, \ldots, \psi_n$ such that $\vdash_\Sigma (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg\phi$, and it follows by an easy argument that $\vdash_\Sigma (\Box\psi_1 \wedge \ldots \wedge \Box\psi_n) \to \Box\neg\phi$. As the reader should check, the formula $(\Box\psi_1 \wedge \ldots \wedge \Box\psi_n) \to \Box(\psi_1 \wedge \ldots \wedge \psi_n)$ is a theorem of every normal modal logic, hence by propositional calculus, $\vdash_\Sigma (\Box\psi_1 \wedge \ldots \wedge \Box\psi_n) \to \Box\neg\phi$. Now, $\Box\psi_1 \wedge \ldots \wedge \Box\psi_n \in w$ (for $\Box\psi_1, \ldots, \Box\psi_n \in w$, and w is an MCS) thus it follows that $\Box\neg\phi \in \phi$. Using $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$, a theorem of every normal modal logic, it follows that $\neg\Diamond\phi \in w$. But this is impossible: w is an MCS containing $\Diamond\phi$. We conclude that v^- is consistent.

Let v be any MCS extending v^-; such extensions exist by Lindenbaum's Lemma. By construction $\phi \in v$. Also, for all formulas ψ, $\Box\psi \in w$ implies $\psi \in v$. Hence, by lemma 2.4.6, $R^\Sigma wv$. $\Box$

Now we can lift the "truth=membership" equation to arbitrary formulas.

Theorem 2.4.8 (**Truth Lemma**). *For any normal modal logic* Σ *and any formula* ϕ, $\Vdash_w^{\mathcal{M}^\Sigma} \phi$ *iff* $\phi \in w$.

Proof. Proof by induction on the degree of ϕ. The base case follows from the definition of V^Σ. The boolean cases follow form theorem 2.4.4, cases (4) and (5). It remains to show the modal case.

The left to right direction is more or less immediate from the definition of R^Σ: $\Vdash_w^{M^\Sigma} \Diamond\psi$ iff there is a v such that $R^\Sigma wv$ and $\Vdash_v^{M^\Sigma} \psi$. By the induction hypothesis this is iff there is a v such that $R^\Sigma wv$ and ψinv and by definition of R^Σ this is if $\Diamond\psi \in w$.

For the right to left direction suppose $\Diamond\phi \in w$. By the equivalences above, it suffices to find a MCS v such that $RLwv$ and $\phi \in v$ – and such v exists by the Existence Lemma. □

Theorem 2.4.9 (**Canonical model theorem**). *Any normal modal logic is strongly complete with respect to its canonical model.*

Proof. Suppose Δ is a consistent set of the normal modal logic Σ. By Lindenbaum's Lemma there is a Σ-MCS Δ^+ extending Δ. By the previous lemma, $\Vdash_{\Delta^+}^{\mathcal{M}^\Sigma} \Delta$. □

We have proved an important theorem about completness. With help of theorem 2.4.3: if we find a class of frames C, to which the canonical model for Σ belongs, by the canonical model theorem and 2.4.3 Σ is strongly complete with respect to C.

As an example of using the just established framework we prove the next result.

Theorem 2.4.10. *K is strongly complete with respect to the class of all frames.*

Proof. By 2.4.3 to prove this result suffices to find for any K-consistent set of formulas Γ a model $\mathcal{M}$ (based on any frame whatsoever) and a state $w \in M$ such that $\Vdash_w^M \Gamma$. This is easy: simply choose $\mathcal{M}$ to be (F^K, V^K), the canonical model of K, and let Γ be any K-MCS extending Γ. By the previous lemma $\Vdash_{\Gamma+}^{(F^K,V^K)} \Gamma$. □

2.4.3 Classical modal logics vs. neighbourhood frames

In this section about soundness and completeness we have considered only the relation between normal modal logics and standard frames.

However, there is a similar approach to soundness and completeness of classical modal logics with respect to neighbourhood frames. Canonical neighbourhood models can be defined and with their help completeness results established. More on this topic can be found in [1]. We won't consider these topics in here, as it is not vital for our work. We will just state some results.

Theorem 2.4.11. *The smallest classical modal logic E is sound and complete with respect to the class of neighborhood frames.*

A corollary of this theorem is that both inclusions of

- the class of normal modal logics under the class classical modal logics
- the class of classical modal logics under the class of all modal logics

are proper. The reason for the first is that not all neighbourhood frames are equivalent to standard frames, and for the second that logics smaller than E are outside the class of the classical modal logics.

2.5 Correspondence theory

Modal formulas can define classes of frames[2] in the following way. A formula defines a class of precisely those frames that validate this formula.

Definition 2.5.1. Let ϕ be a modal formula, and C a class of frames. We say that ϕ *defines* C if

$$\mathcal{F} \in C \text{ iff } \Vdash^{\mathcal{F}} \phi. \qquad \dashv$$

Modal formulas can't define all classes of frames.

Proposition 2.5.1. *There is a frame class such that there is no modal formula defining this class.*

Proof. See example 3.15 in [2]. □

However, we will turn to frame classes that are modally definable. We can define a class of frames not only using a modal formula, but also by setting a restriction on the accessibility relation of the frame. For example, the class of transitive frames, the class of reflexive frames. This condition (or frame property) can be expressed using a first- or second-order formula. For expressing conditions on frames, we can use one of following languages.

Definition 2.5.2. The *first-order frame language* is the first order language that has the identity symbol = together with binary relation symbols $R_\Box$ and $R_\Diamond$. We denote this language by $\mathcal{L}_1$. We also call it the *first-order correspondence language.*

The *monadic*[3] *second-order frame language* is the monadic second-order language obtained by extending $\mathcal{L}_1$ with a collection of monadic predicate variables indexed over propositional variables. We denote it by $\mathcal{L}_2$. We often call it *second-order frame language* or *second-order correspondence language.* ⊣

[2]In this whole section we consider standard frames.

[3]i.e. unary

Sometimes we can find classes of frames that can be equally defined using modal formulas and via setting a restriction on the accessibility relation. For example, the class of transitive frames is definable by schema 4: $\Box\phi \rightarrow \Box\Box\phi$. We call formulas 4 and $\forall u, v, w : Ruv \wedge Rvw \rightarrow Ruw$ local frame correspondents.

Definition 2.5.3. Let ϕ be a modal formula and $\alpha(x)$ a formula in the corresponding first- or second-order frame language (x is supposed to be the only free variable in α). Then we say that ϕ and $\alpha(x)$ are *(local) frame correspondents* of each other if the following holds, for any frame $\mathcal{F}$ and any state w of $\mathcal{F}$:

$$\Vdash^{\mathcal{F}}_{w}\phi \text{ iff } \Vdash^{\mathcal{F}} \alpha[w] \qquad \dashv$$

The second-order frame language is powerful; it can describe all frames that modal formulas can define.

Proposition 2.5.2. *Modal formulas standardly correspond to second-order frame conditions.*

To prove this proposition, we will introduce the standard translation.

Definition 2.5.4. Let x be a first-order variable. The *standard translation* ST_x taking modal formulas to first-order formulas in $\mathcal{L}_1$ is defined as follows

$$\begin{array}{rcl} ST_x(p) & = & Px, \\ ST_x(\bot) & = & x \neq x, \\ St_x(\neg\phi) & = & \neg ST_x(\phi), \\ St_x(\phi \vee \psi) & = & ST_x(\phi) \vee ST_x(\psi), \\ ST_x(\Box\phi) & = & \forall y(Rxy \rightarrow ST_y(\phi)), \end{array}$$

where y is a fresh variable. $\dashv$

Note that the standard translation is simply restating the definition of satisfaction in terms of first-order logic.

Theorem 2.5.3. *Let ϕ be a formula. Then for any frame $\mathcal{F}$ and any state w:*

$$\Vdash^{\mathcal{F}}_{w}\phi \textit{ iff } \mathcal{F} \vDash \forall P_1 \ldots P_n ST_x(\phi)[w]$$

$$\Vdash^{\mathcal{F}} \phi \textit{ iff } \mathcal{F} \vDash \forall P_1 \ldots P_n ST_x(\phi)$$

Here, the second-order quantifiers bind second-order variables $P_1, \ldots, P_n$ corresponding to the propositional variables p_i occurring in ϕ.

Proof. By induction on the construction of formula ϕ. $\Box$

But sometimes, even the first-order frame logic is sufficient to describe the frame conditions.

Proposition 2.5.4. *There are modal formulas locally corresponding to first-order frame logic formulas.*

Proof. For example $\Box\phi \to \phi$ locally corresponds to $\forall u, v, w : Ruv \wedge Rvw \to Ruw$. We leave the proof as an exercise for the reader.

We present an example in section about distribution of modal operator over disjunction. (see theorem 3.1.4) $\Box$

However, the first-order frame logic can't describe all frames modal formulas can define.

Proposition 2.5.5. *There are modal formulas corresponding to no first-order frame logic formula.*

Proof. We present an example of such formula in section about distribution of modal operator over disjunction. (see proposition 3.1.1) $\Box$

The question is, what is the class of modal formulas, that have first-order local correspondents. A sufficient condition on modal formulas gives the Sahlqvist's Theorem, named after a Norwegian mathematician Henrik Sahlqvist. This theorem is also called Sahlqvist's Correspondence Theorem, since there is also Sahlqvist's Completeness Theorem (see section 3.1.3).

First, we will need to define certain syntactical classes of formulas.

Definition 2.5.5. An occurrence of a propositional variable p is a *positive occurrence* if it is in the scope of an even number of negation signs; it is a *negative occurrence* if it is in scope of an odd number of negation signs. A modal formula ϕ is *positive in* p (*negative in* p) if all occurrences of p in ϕ are positive (negative). A formula is called *positive* (*negative*) if it is in positive (negative) in all propositional variables occurring in it. $\dashv$

Definition 2.5.6. *Boxed atom* is a formula of the form $\Box \ldots \Box p$, $(k \geq 0)$ where p is a propositional variable. $\dashv$

Definition 2.5.7. We call a formula *quasi-atomic* if it is a propositional variable or a boxed atom. We will denote the set of all quasi-atomic formulas in Φ^{ML} as Φ^*. $\dashv$

Definition 2.5.8. A *Sahlqvist antecedent* is a formula built up from $\top, \bot$, boxed atoms and negative formulas, using $\wedge, \vee$ and $\Diamond$. A *Sahlqvist implication* is an implication $\phi \to \psi$ in which ψ is positive and ϕ is a Sahlqvist antecedent.

A *Sahlqvist formula* is a formula that is built up from Sahlqvist implications by freely applying boxes and conjunctions, and by applying disjunctions only between formulas that do not share any propositional variables. $\dashv$

Theorem 2.5.6 (Sahlqvist's Theorem). *Let χ be a Sahlqvist formula. Then χ locally corresponds to a first-order formula $c_\chi(x)$ on frames. Moreover, c_χ is effectively computable from χ.*

Proof. Proof is lengthy and quite complicated. It is not essential for the purposes of our work. The proof can be found in detail in [2], theorem 3.49.

However, from the proof we can extract the reason why Sahlqvist formulas have first-order correspondents. Syntactically, the Sahlqvist fragment forbids universal operators to take scope over existential or disjunctive connectives in the antecedent. Semantically, this guarantees that we will always are able to find a unique minimal valuation that makes the antecedent true. This is what ensures that Sahlqvist formulas have first-order correspondents. □

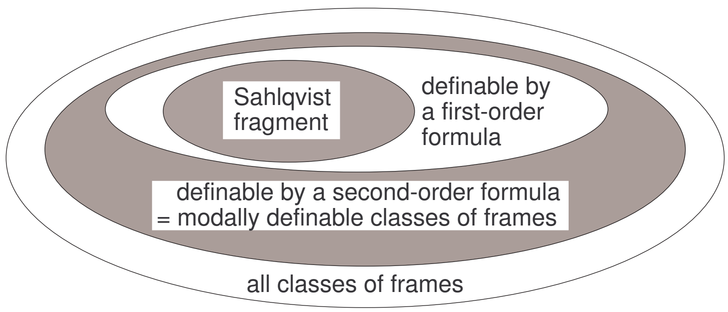

Figure 2.6: Definable classes of frames and Sahlqvist fragment

The class of Sahlqvist formulas is called the Sahlqvist fragment. It does not contain all modal formulas with first order correspondents.

Proposition 2.5.7. *There are non-Sahlqvist formulas that define first-order conditions.*

Proof. We present an example of such formula in section about distribution of modal operator over disjunction. (see theorem 3.1.4) □

The Sahlqvist fragment can be extended further. Nevertheless, it is a good compromise between the demands of simplicity and generality. The Sahlqvist fragment cannot be further extended just by dropping some of the restrictions in the definition of the Sahlqvist formula. Forbidden combinations easily lead to modal formulas, that have no first-order correspondent (see proposition 3.1.1).

Unfortunately, there is one more negative result. Chagrova's theorem tells us that:

Theorem 2.5.8. *It is undecidable whether a modal formula has a first order equivalent.*

Proof. The proof can be found in [10]. □

Chapter 3

Distribution of the modal operator over disjunction

We call formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ *distribution of modal operator over disjunction* and denote it Distr∨.

In the introduction of this work (see 1.2) we have seen the motivation for investigating properties of Distr∨.

In this section we consider this formula with respect to two types of modal logics, normal and non-normal.

In normal logics, we are looking for a class of frames defined by this formula. We call this class a class of deterministic frames. Then we show soundness and strong completeness of logic KDistr∨, the smallest normal logic providing KDistr∨, with respect to the class of deterministic frames. In spite of the fact that this formula is not in the form that automatically guarantees correspondence and strong completeness, we manage to prove these.

In non-normal logics we will also show a class of neighbourhood frames defined by this formula. Then we will explore the property of neighbourhood frames defining this class – non-emergence – namely how to test frames for this property.

We will also look at equivalent formulas and a formula representing the converse principle.

At the end of each part we try to sketch some applications of our results.

3.1 In normal modal logics

3.1.1 Frame correspondence

In section 2.5, we considered the Sahlqvist fragment. Every Sahlqvist formula has a first-order frame correspondent. However, being Sahlqvist is only a sufficient condition. There are non-Sahlqvist formulas which correspond to a first-order frame property.

Sahlqvist formulas satisfy a certain syntatically defined property. These formulas cannot contain disjunction in the scope of the modal operator. The reason is, as we will see, that this forbidden combination often leads to modal formulas that have no first-order correspondent.

We present an example of such formula.

Proposition 3.1.1. *Formula* $\Box(\phi \vee \psi) \to \Diamond(\Box\phi \vee \Box\psi)$ *has no first-order frame correspondent.*

Proof. To prove it, we will show that it violates the Löwenheim–Skolem Theorem (see 3.1.2).

Consider a standard frame $\mathcal{F} = (W, R)$ where

$$W = \{w\} \cup \{v_{n_i} \mid n \in \mathbb{N};\ i \in \{0,1\}\} \cup \{z_f \mid f : \mathbb{N} \to \{0,1\}\}$$

and

$$R = \{(w, v_{n_i}), (v_{n_i}, v_{n_j}) \mid n \in \mathbb{N};\ i, j \in \{0,1\}\}$$
$$\cup\{(w, z_f), (z_f, v_{n_{f(n)}}) \mid f : \mathbb{N} \to \{0,1\};\ n \in \mathbb{N}\}$$

In a diagram:

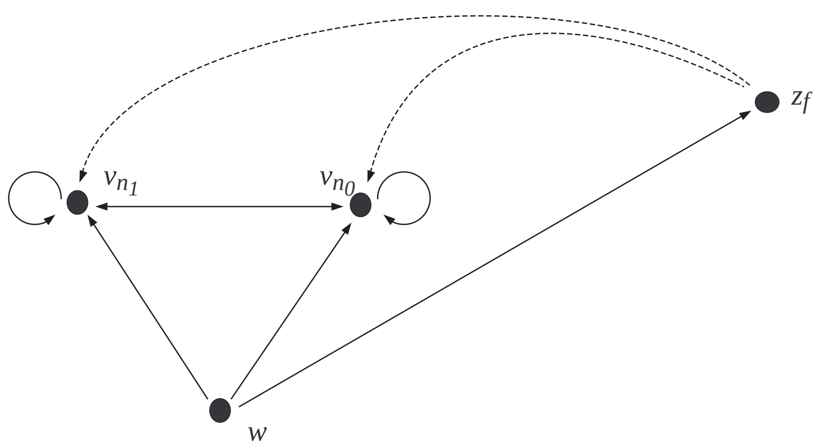

Figure 3.1: A frame validating a formula that has no first-order frame correspondent

We first prove that $\Vdash^{\mathcal{F}}_{w} \Box(\phi \vee \psi) \to \Diamond(\Box\phi \vee \Box\psi)$. If $\Vdash^{(\mathcal{F},V)}_{w} \Box(\psi \vee \psi)$, then one of the following cases appears:

- For some $n \in \mathbb{N}$, both $\Vdash^{(\mathcal{F},V)}_{v_{n_0}} \psi$ and $\Vdash^{(\mathcal{F},V)}_{v_{n_1}} \psi$, that means $\Vdash^{(\mathcal{F},V)}_{v_{n_0}} \Box\psi$.
  Thus, $\Vdash^{(\mathcal{F},V)}_{w} \Diamond(\Box\psi \vee \Box\psi)$ holds.

- For each $n \in \mathbb{N}$, $\Vdash_{v_{n_0}}^{(\mathcal{F},V)} \psi$ or $\Vdash_{v_{n_1}}^{(\mathcal{F},V)} \psi$ – then there is $f : N \to \{0,1\}$ such that $\Vdash_{v_{n_{f(n)}}}^{(\mathcal{F},V)} \psi$ for all $n \in N$. Hence $\Vdash_{z_f}^{(\mathcal{F},V)} \Box\psi$, and then $\Vdash_{w}^{(\mathcal{F},V)} \Diamond(\Box\psi \vee \Box\psi)$ holds.

In order to show that $\Box(\psi \vee \psi) \to \Diamond(\Box\psi \vee \Box\psi)$ does not define a first-order frame condition, let us view the frame $\mathcal{F}$ as a first-order model with domain W. The set W contains uncoutably many points, for the set of functions $f : \mathbb{N} \to \{0,1\}$ indexing the z-points is uncountable. By the downward Löwenheim–Skolem Theorem there must be a countable elemetary submodel $\mathcal{F}'$ of $\mathcal{F}$ whose domain W' contains w, and each v_{n_0} and v_{n_1}. As W is uncountable and W' countable, there must be a mapping $g : \mathbb{N} \to \{0,1\}$ such that z_g does not belong to W'. Now if our formula was equivalent to a first-order formula it would be valid on $\mathcal{F}'$, since the Löwenheim–Skolem Theorem tells us that $\mathcal{F}$ and $\mathcal{F}'$ are elementarily equivalent. But we will show that our formula is not valid on $\mathcal{F}'$, hence it cannot be equivalent to a first-order formula.

Let V' be a valuation on $\mathcal{F}'$ such that $V'(p) = \{v_{n_{g(n)}} \mid n \in \mathbb{N}\}$ and $V'(q) = W \backslash V'(p)$, where g is mapping described above. Trivially $\Vdash_{w}^{(\mathcal{F}',V')} \Box(p \vee q)$. But since for each $n \in \mathbb{N}$ either $\Vdash_{v_{n_0}}^{(\mathcal{F}',V')} p$ or $\Vdash_{v_{n_1}}^{(\mathcal{F}',V')} p$, then neither $\Box p$ nor $\Box q$ is true at any v_{n_i}. Moreover, $\Box p$ is true at any $z_h \in W'$, for if it was, v_h would be the same as v_g but this is not in W'. And neither $\Box q$ is true at any $z_h \in W'$, for its presence in W' would imply presence of z_g as well, since $V(p) \cap V(q) = \emptyset$ and $V(p) \cup V(q) = W'$. That means $\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$ is not true at $w \in \mathcal{F}'$. □

Definition 3.1.1. Two first-order models are *elementarily equivalent*, if every formula true in one is true in the other and vide versa. A *submodel* is a subset of a model.

For definitions and basic model-theoretic concepts please refer to [2], Appendix A, page 494; or any book about first-order logic. ⊣

Theorem 3.1.2 (**Löwenheim–Skolem theorem**). *Let $\mathfrak{A}$ be a model of cardinality α and let the $|\mathcal{L}| \leq \beta \leq \alpha$, where $|\mathcal{L}|$ is the number of non-logical symbols in the first-order language $\mathcal{L}$. Then $\mathfrak{A}$ has an elementary submodel of cardinality β. Furthermore, given any set $X \subseteq A$ of cardinality $\leq \beta$, $\mathfrak{A}$ han an elementary submodel of cardinality β which contains X.*

The formula we are investigating in this chapter – $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ – is non-Sahlqvist, because it contains a disjunction in the scope of $\Box$. That means Sahlqvist's Theorem doesn't ensure a first-order property of class of frames it defines. As we have seen, formulas with modal operator over disjunction sometimes fail to have first-order correspondents. However, this is not the case of $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$. Formula $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ defines a class of deterministic frames.

Proposition 3.1.3. *Formula $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ is not valid in the class of all frames.*

Proof. Consider $\mathcal{M} = (W, R, V)$ such that $W = w, v_1, v_2$, Rwv_1 and Rwv_2, $V(p) = \{v_1\}$ and $V(q) = \{v_2\}$. In both v_1, v_2, $p \vee q$ is true, so $\Vdash_{w}^{\mathcal{M}} \Box(p \vee q)$. But obviously $\nVdash_{w}^{\mathcal{M}} \Box p \vee \Box q$. Thus, $\nVdash_{w}^{\mathcal{M}} \Box(p \vee q) \to (\Box p \vee \Box q)$. □

Definition 3.1.2. We will call a standard frame $\mathcal{F} = (W, R)$ *deterministic*, iff for all w, v_1, v_2 in W:

$$\text{if } Rwv_1 \text{ and } Rwv_2, \text{ then } v_1 = v_2. \qquad \dashv$$

Theorem 3.1.4. *Frame* $\mathcal{F} = (W, R)$ *is deterministic iff* $\Vdash^{\mathcal{F}} \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$.

Proof. For the left to right direction. Suppose by contradiction, that frame $\mathcal{F}$ is deterministic but $\nVdash^{\mathcal{F}} \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$. Then there is a model $\mathcal{M}$ based on $\mathcal{F}$ and a w such, that $\nVdash_w^{\mathcal{M}} \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$.

That means $\Vdash_w^{\mathcal{M}} \Box(\phi \vee \psi)$ and $\nVdash_w^{\mathcal{M}} (\Box\phi \vee \Box\psi)$. The latter by definition entails $\nVdash_w^{\mathcal{M}} (\Box\phi)$ and $\nVdash_w^{\mathcal{M}} (\Box\psi)$.

From the last two we know that $\exists v_1, Rwv_1 \nVdash_{v_1}^{\mathcal{M}} \phi$ and $\exists v_2, Rwv_2 \nVdash_{v_2}^{\mathcal{M}} \psi$. But since $\mathcal{F}$ is deterministic, $v_1 = v_2$. That means $\nVdash_{v_1}^{\mathcal{M}} \phi$ and $\nVdash_{v_1}^{\mathcal{M}} \psi$. Thus in v_1 the disjunction of ϕ and ψ is false. But then $\nVdash_w^{\mathcal{M}} \Box\phi \vee \psi$, and we get a contradiction with what we supposed.

For right to left suppose, again by contradiction, that $\Vdash^{\mathcal{F}} \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ but $\mathcal{F}$ is not deterministic. Thus

$$\exists w, v_1, v_2 \in W: \quad Rwv_1 \text{ and } Rwv_2 \text{ and } v_1 \neq v_2$$

Then there is a valuation V such that $V(p) = \{v \in W \mid Rwv; v \neq v_2\}$ and $V(q) = \{v_2\}$.

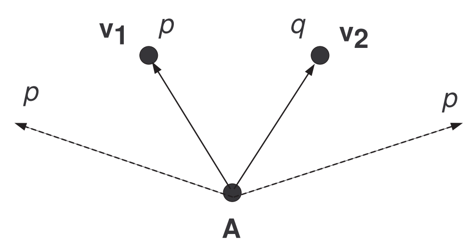

Figure 3.2: Valuation V in the proof that a frame is deterministic

Thus $\Vdash_w^{(\mathcal{F},V)} \Box(p \vee q)$ but $\nVdash_w^{(\mathcal{F},V)} \Box p$ nor $\nVdash_w^{(\mathcal{F},V)} \Box q$, which is a contradiction with validity of formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$. $\Box$

3.1.2 Equivalent formulas

Now we introduce formulas, which are equivalent to the distribution of modal operator over disjunction. In every normal modal logic, $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ is equivalent to formulas: $\Box\phi \vee \Box\neg\phi$, $\Diamond\phi \rightarrow \Box\phi$, $(\Diamond\phi \wedge \Diamond\psi) \rightarrow \Diamond(\phi \vee \psi)$, $\neg(\Diamond\phi \wedge \Diamond\neg\phi)$ and $\Diamond\phi \rightarrow \phi$. We will prove the equivalence below.

As a result of the equivalence we have that each of these formulas defines the class of transitive frames.

Theorem 3.1.5. *Let $\phi[\psi/\psi']$ be any formula that results from ϕ by replacing zero or more occurences of ψ in ϕ by ψ'. Following inference rule is provided by every normal modal logic*

$$REP. \quad \frac{\psi \leftrightarrow \psi'}{\phi \leftrightarrow \phi[\psi/\psi']}$$

Proof. We leave the proof to the reader. Hint: by induction on the complexity of ϕ. Proved in [1], theorem 4.7. $\Box$

Theorem 3.1.6. *Following formulas are all equivalent.*

1. $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$
2. $\Box\phi \vee \Box\neg\phi$
3. $\Diamond\phi \rightarrow \Box\phi$
4. $(\Diamond\phi \wedge \Diamond\psi) \rightarrow \Diamond(\phi \vee \psi)$
5. $\neg(\Diamond\phi \wedge \Diamond\neg\phi)$
6. $\Diamond\phi \rightarrow \phi$

Proof.

1. (1)⇒ (2)

1.	$\phi \vee \neg\phi$	PL
2.	$\Box(\phi \vee \neg\phi)$	1, RN
3.	$\Box(\phi \vee \neg\phi) \rightarrow (\Box\phi \vee \Box\neg\phi)$	(1)
4.	$\Box\phi \vee \Box\neg\phi$	2, 3, MP

2. (2)⇒ (3)

1.	$\Box\neg\phi \vee \Box\phi$	(2)
2.	$\neg\Diamond\neg\neg\phi \vee \Box\phi$	1, Df$\Diamond$
3.	$\neg\Diamond\phi \vee \Box\phi$	REP, PL
4.	$\Diamond\phi \rightarrow \Box\phi$	3, PL

3. (3)⇒ (1)

1.	$\Box(\phi \vee \psi) \rightarrow \Box(\neg\phi \rightarrow \psi)$	PL, REP
2.	$\Box(\neg\phi \rightarrow \psi) \rightarrow (\Box\neg\phi \rightarrow \Box\psi)$	K
3.	$\Box(\phi \vee \psi) \rightarrow (\Box\neg\phi \rightarrow \Box\psi)$	1, 2, PL
4.	$\Box(\phi \vee \psi) \rightarrow (\Diamond\phi \vee \Box\psi)$	Df$\Diamond$, REP
5.	$\Diamond\phi \rightarrow \Box\phi$	(3)
6.	$(\Diamond\phi \vee \Box\psi) \rightarrow (\Box\phi \vee \Box\psi)$	5, PL
7.	$\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$	4, 6, PL

The last three formulas are so called duals of the first three ones. We leave it as an exercise for the reader to prove their equivalence to the other ones. $\Box$

3.1.3 Soundness and completeness

Sahlqvist fragment provides one more useful property. Given a set of Sahlqvist axioms Σ, the logic $K\Sigma$ is strongly complete with respect to the class of frames defined by Σ. This result, called Sahlqvist's Completeness Theorem connects correspondence with completeness. Note that this is a very useful, because most of commonly used axioms are Sahlqvist.

As we know, our formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ is not Sahlqvist. However, we proved that it defines a class of deterministic frames, which is a first-order property. Similarly, we cannot use the Sahlqvist's Completeness Theorem for showing that the normal modal logic $KDistr\vee$ is sound and complete with respect to the class of deterministic frames.

However, we can still prove that $KDistr\vee$ is sound and strongly complete with respect to this class of frames. We will show it as follows.

Theorem 3.1.7. *$KDistr\vee$ is sound and strongly complete with respect to the class of deterministic frames.*

Proof. Soundness. Axioms Df$\Diamond$ and K are valid in any class of frames. Axiom Distr$\vee$ is valid in this class of frames by the left-to-right direction of theorem 3.1.4. Inference rules MP, generalization, and uniform substitution preserve validity on the this class of frames by 2.4.1.

Strong completeness. Given a $KDistr\vee$-consistent set of formulas Γ, it suffices to find a model $(\mathcal{F}, V)$ and a state w in this model such that

1. $\Vdash_w^{(\mathcal{F},V)} \Gamma$,

2. $\mathcal{F}$ is deterministic.

Let $\mathcal{M}^{KDistr\vee} = (W^{KDistr\vee}, R^{KDistr\vee}, V^{KDistr\vee})$ be the canonical model for $KDistr\vee$ and let Γ^+ be any $KDistr\vee$-MCS extending Γ. By theorem 2.4.8, $\Vdash_{\Gamma^+}^{\mathcal{M}^{KDistr\vee}} \Gamma$.

Now we will show, that $(W^{KDistr\vee}, R^{KDistr\vee})$ is deterministic. Suppose u, v are states in this frame such that $R^{KDistr\vee}uv$. Suppose that $\phi \in v$. As $R^{KDistr\vee}uv$, $\Diamond\phi \in u$. As $KDistr\vee$ is a normal modal logic containing KDistr$\vee$, by theorem 3.1.6 it also contains formula $\Diamond\phi \rightarrow \Box\phi$. And since u is a $KDistr\vee$-MCS, we have that $\Diamond\phi \rightarrow \Box\phi \in u$. Since also $\Diamond\phi \in u$, then by modus ponens also $\Box\phi \in u$. This means, that for every state w such that $R^{KDistr\vee}uw$, $\phi \in w$. And since all states w accessible from u are MCSs and formula ϕ was arbitrary and it must be contained in all states w, we get that there is only one state w accessible from u. Hence, frame $(W^{KDistr\vee}, R^{KDistr\vee})$ is deterministic. $\Box$

In fact this proof establishes something more general that the theorem claims: That canonical frame of *any* normal modal logic Σ containing $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ is deterministic. The proof works because all MCSs in the canonical frame contain the axiom Distr$\vee$. Thus, the canonical frame of any extension of $KDistr\vee$ is deterministic.

3.1.4 Applications

The message of theorem 3.1.7 and the following remark is following. Every time a normal modal logic contains formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ or any of its equivalents (see section 3.1.2), the frame validating all formulas of this logic will be deterministic.

This gives rise to applications. From the semantic perspective, the condition of being determinic, i.e. for all w, v_1, v_2 in W:

$$\text{if } Rwv_1 \text{ and } Rwv_2, \text{ then } v_1 = v_2$$

is useful whenever reasoning about a problem whose domain is deterministic, e.g. programs and other deterministic phenomenons. To make any model of logic deterministic, we include the formula Distr$\vee$.

On the other hand, if we come to a phenomenon that we know can intuitively be described by formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$, we immediately get the nature of the domain. It must be deterministic.

3.2 In non-normal modal logics

Now we turn to non-normal modal logics.

3.2.1 Correspondence

As in section 3.1.1 in case of normal modal logics and standard frames, we investigate the formula Distr$\vee$ with repect to non-normal modal logics and neighbourhood frames.

We will show that the formula $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ is not valid on the class of all neighbourhood frames.

Then we will give a semantic property of neighbourhood frames. A class of frames with this property will prove to be precisely the class of frames validating formula Distr$\vee$.

We will start with the fact that Distr$\vee$ is not valid in the class of all neighbourhood frames. This is not surprising, since it wasn't valid even in the class of all standard frames.

Proposition 3.2.1. *There is a neighbourhood model $\mathfrak{M}$ and a state w such that an instance of schema Distr$\vee$ is in this state not satisfied, i.e. $\not\models^{\mathfrak{M}}_{w} \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$*

Proof. Consider model $\mathfrak{M} = (W, N, V)$, where $W = \{w_1, w_2, w_3\}$, $N(w_1) = \{\{w_2, w_3\}\}$, $N(w_2) = N(w_3) = \emptyset$ and $V(p) = \{w_2\}, V(q) = \{w_3\}$.

In this model $\models^{\mathfrak{M}}_{w_1} \Box(p \vee q)$, because $\|p \vee q\|^{\mathfrak{M}} = \{w_2, w_3\}$ so $\|p \vee q\|^{\mathfrak{M}} \in N(w_1)$. Furthermore, $\not\models^{\mathfrak{M}}_{w_1} \Box p$, because $\|p\|^{\mathfrak{M}} = \{w_2\} \notin N(w_1)$ and also $\not\models^{\mathfrak{M}}_{w_1} \Box q$, because $\|q\|^{\mathfrak{M}} = \{w_3\} \notin N(w_1)$. Hence $\not\models^{\mathfrak{M}}_{w_1} \Box p \vee \Box q$.

We showed that $\not\models^{\mathfrak{M}}_{w_1} \Box(p \vee q) \rightarrow (\Box p \vee \Box q)$ □

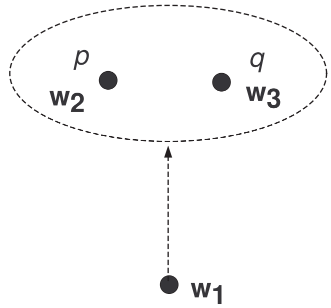

Figure 3.3: A neighbourhood model falsifying Distr∨

Now we will define a property of neighbourhood frames, that will ensure validity of Distr∨.

Definition 3.2.1. We will call a neighbourhood frame $\mathfrak{F} = (W, N)$ *non-emergent*, iff for every world w and for all sets of worlds $A, B \subseteq W$:

$$\text{if } A \cup B \in N(w) \text{ then } A \in N(w) \text{ or } B \in N(w). \qquad \dashv$$

Now we show that non-emergence is a sufficient condition for a neighborhood frame to validate formula $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$.

Lemma 3.2.2. *If a neighbourhood frame $\mathfrak{F}$ is non-emergent, then $\vDash^{\mathfrak{F}} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$.*

Proof. Consider non-emergent neighbourhood frame $\mathfrak{F} = (W, N)$. We will show, that $\vDash^{\mathfrak{F}} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$.

Suppose by contradiction that there is a model $\mathfrak{M}$ based on $\mathfrak{F}$, state w and formulas ϕ', ψ' such that $\nvDash^{\mathfrak{M}}_{w} \Box(\phi' \vee \psi') \to (\Box\phi' \vee \Box\psi')$.

That means that $\vDash^{\mathfrak{M}}_{w}\Box(\phi' \vee \psi')$ and $\nvDash^{\mathfrak{M}}_{w} \Box\phi' \vee \Box\psi'$. From the former we have that $\|\phi' \vee \psi'\|^{\mathfrak{M}} = (\|\phi'\|^{\mathfrak{M}} \cup \|\psi'\|^{\mathfrak{M}}) \in N(w)$. Since $\mathfrak{F}$ is non-emergent, $\|\phi'\|^{\mathfrak{M}} \in N(w)$ or $\|\phi'\|^{\mathfrak{M}} \in N(w)$. This means $\vDash^{\mathfrak{M}}_{w'}\Box\phi$ or $\vDash^{\mathfrak{M}}_{w'}\Box\psi'$, thus $\vDash^{\mathfrak{M}}_{w}\Box\phi' \vee \Box\psi'$.

But this is a contradiction with the assumption that $\nvDash^{\mathfrak{M}}_{w} \Box\phi' \vee \Box\psi'$. $\Box$

Non-emergence is a necessary condition too.

Lemma 3.2.3. *If $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ is valid in the neighbourhood frame $\mathfrak{F}$, then $\mathfrak{F}$ is non-emergent.*

Proof. Suppose by contradiction that neighbourhood frame $\mathfrak{F} = (W, N)$ validates Distr∨, i.e. $\vDash^{\mathfrak{F}} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ but $\mathfrak{F}$ doesn't satisfy the condition of non-emergence.

Then there is a state w' and exist $A, B \in N(w')$ such that

$$A \cup B \in N(w') \text{ and } A \notin N(w') \text{ and } B \notin N(w')$$

Since $\models^{\mathfrak{F}} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$, for all neighbourhood models $\mathfrak{M}$ based on neighbourhood frame $\mathfrak{F}$ and all worlds w, and all formulas ϕ, ψ, $\models^{\mathfrak{M}}_{w} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$.

Then also for a model $\mathfrak{M}' = (\mathfrak{F}, V')$ such that $V'(p) = A$ and $V'(q) = B$ we have $\models^{\mathfrak{M}'}_{w'} \Box(p \vee q) \to (\Box p \vee \Box q)$.

But we know, that

- $\models^{\mathfrak{M}'}_{w'} \Box(p \vee q)$ since $\|p \vee q\|^{\mathfrak{M}'} = \|p\|^{\mathfrak{M}'} \cup \|q\|^{\mathfrak{M}'} = A \cup B \in N(w')$
- $\not\models^{\mathfrak{M}'}_{w'} \Box p \vee \Box q$ since $\not\models^{\mathfrak{M}'}_{w'} \Box p$ and $\not\models^{\mathfrak{M}'}_{w'} \Box q$,
  because $\|p\|^{\mathfrak{M}'} = A \notin N(w')$ and $\|p\|^{\mathfrak{M}'} = B \notin N(w')$

Hence, $\not\models^{\mathfrak{M}'}_{w'} \Box(p \vee q) \to (\Box p \vee \Box q)$ which is a contradiction. □

Hence, we have proved that, using terms from the correspondence theory:

Theorem 3.2.4. *Distr$\vee$ defines the class of non-emergent neighbourhood frames.*

Note that this is a first order property.

3.2.2 Normal as a special case of non-normal

Since essentially every normal modal logic is a special case of non-normal modal logics (see figure 2.2.2), with help of the soundness and completness theorems (see section 2.4) we know, that every standard frame is equivalent to a special case of neighbourhood frames.

To put this precise we introduce the notion of augmentation.

Definition 3.2.2. We will call a neighbourhood frame $\mathfrak{F} = (W, N)$ *supplemented*, iff for every world w and for all sets of worlds $A, B \subseteq W$:

$$\text{if } A \cap B \in N(w) \text{ then } A \in N(w) \text{ and } B \in N(w). \qquad \dashv$$

Definition 3.2.3. A neighbourhood frame $\mathfrak{F} = (W, N)$ is called *augmented* iff it is supplemented and for every world w in it, $\cap N(w) \in N(w)$. $\dashv$

Thus in an augmented neighbourhood frame each $N(w)$ contains a smallest proposition, the set comprising just those worlds that are members of every proposition in $N(w)$. We can augmented neighbourhood frames equally characterize as follows.

Proposition 3.2.5. *$\mathfrak{F}$ is augmented iff for every w and X*

$$X \in N(w) \textit{ iff } \cap N(w) \subseteq X$$

Proof. If $\mathfrak{F}$ is augmented and $\cap N(w) \subseteq X$, then by supplementation $X \in N(w)$. On the other hand, suppose $\mathfrak{F}$ satisfies the condition. Then if $X \subseteq Y$ and $X \in N(w)$ it follows that $\cap N(w) \subseteq Y$, which means that $Y \in N(w)$. So $\mathfrak{F}$ is supplemented. Also, $\cap N(w) \subseteq \cap N(w)$, since $\cap N(w) \subseteq \cap N(w)$. Hence, the neighbourhood model is augmented. □

The relationship between standard frames and neighbourhood frames is, that a standard frame is essentially an augmented neighbourhood frame.

Theorem 3.2.6. *For every standard model $\mathcal{M} = (W, R, V)$ there is a pointwise equivalent augmented neighbourhood mode $\mathfrak{M} = (W, N, V)$ and vice versa.*

Proof. Let $\mathcal{M}$ be a standard model and define the neighbourhood model $\mathfrak{M}$ by:

$$X \in N(w) \text{ iff } \{v \in W \mid Rwv\} \subseteq X$$

for every $w \in W$ and every $X \subseteq W$. Then $\cap N(w) = \{v \in W \mid Rwv\}$ for each $w \in W$. So $\mathfrak{M}$ satisfies condition of augmentation.

The proof that $\mathcal{M}$ and $\mathfrak{M}$ are pointwise equivalent, i.e. that a world verifies the same formulas in each model, is by induction on the complexity of a formula ϕ. The only interesting case is when $\phi = \Box\psi$:

$$\begin{array}{lll} \Vdash_w^{\mathcal{M}} \Box\psi & \text{iff} & \text{for every } v \in W \text{ such that } Rwv, \Vdash_v^{\mathcal{M}} \psi \\ & \text{iff} & \{v \in W \mid Rwv\} \subseteq \|\psi\|^{\mathcal{M}} \\ & \text{iff} & \|\psi\|^{\mathfrak{M}} \in N(w) \\ & \text{iff} & \vDash_w^{\mathfrak{M}} \Box\psi \end{array}$$

For the other direction, let $\mathfrak{M}$ be an augmented neighbourhood model and define the standard model $\mathcal{M}$ by: Rwv iff $v \in \cap N(w)$ for every w and v in W.

As before, necessitation is the only case of interest in the inductive proof that the models are pointwise equivalent.

$$\begin{array}{lll} \vDash_w^{\mathfrak{M}} \Box\psi & \text{iff} & \|\psi\|^{\mathfrak{M}} \in N(w) \\ & \text{iff} & \cap N(w) \in \|\psi\|^{\mathfrak{M}} \\ & \text{iff} & \text{for every } v \in W \text{ such that } Rwv, \Vdash_v^{\mathcal{M}} \psi \\ & \text{iff} & \Vdash_w^{\mathcal{M}} \Box\psi \end{array}$$

This completes the proof of the theorem. □

This theorem and its (constructive) proof gives us a new tool to analyze properties of standard models. Now, from neighbourhood models possessing a property we can turn to standard models with the same property. These are augmented neighbourhood models.

In the section 3.1.1 we showed that that formula Distr$\vee$ defines the class of deterministic standard frames. Then, in section 3.2.1 we showed that it defines the class of non-emergent neighbourhood frames. If we were right, these classes should coincide on standard frames, i.e. every non-emergent augmented frame should be pointwise equivalent to a deterministic standard one.

Standard frames validating formula $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ are augmented neighbourhood frames satisfying the condition of non-emergence.

If we take an augmented and non-emergent frame, that is for every w and every X, A, B we have

- if $A \cup B \in N(w)$ then $A \in N(w)$ or $B \in N(w)$
- $X \in N(w)$ iff $\cap N(w) \subseteq X$

Combining these, we get for all A, B and w

$$A \cup B \supseteq \cap N(w) \text{ then } A \supseteq \cap N(w) \text{ or } B \supseteq \cap N(w)$$

or, equivalently, for all A, B and w

$$\text{if } A \subset \cap N(w) \text{ and } B \subset \cap N(w) \text{ then } A \cup B \subset \cap N(w)$$

Suppose, $\cap N(w)$ had at least two elements, denote two of them a_1 and a_2. Consider sets $A_1 = \cap N(w) \setminus \{a_1\}$ and $A_2 = \cap N(w) \setminus \{a_2\}$. For both $i = 1, 2$ $A_i \subset \cap N(w)$ holds, but $A_1 \cup A_2 = \cap N(w) \not\subset \cap N(w)$. In other words, whenever $\cap N(w)$ contains at least two elements, condition doesn't hold.

Let's check if the condition holds in $\cap N(w)$ possessing just one element. If $\cap N(w) = \{a\}$. Then subsets A of $\cap N(w)$ are only $\emptyset$ and $\{a\}$ the only proper subset being empty set. Thus, condition holds, because $\emptyset \cup \emptyset = \emptyset \subset \{a\}$.

If $\cap N(w) = \emptyset$, then condition holds trivially.

We have shown, that in augmented non-emergent frames, the $\cap N(w)$ for all w must contain at most one world.

Hence, we have just proved following theorem

Theorem 3.2.7. *To every augmented non-emergent neighbourhood frame there exist a pointwise-equivalent deterministic standard frame and vice versa.*

3.2.3 Non-emergence

Formula Distr$\vee$ defines a property of neighbourhood frames, namely non-emergence. This condition is intuitively hard to imagine. For potential applications we need to test the neighbourhood whether it is non-emergent or generate non-emergent neighbourhoods. We investigate this condition in some detail in this section.

First, we repeat the definition of non-emergence, this time more abstractly.

Definition 3.2.4. Consider a non-empty set W. We call *neighbourhood* a family of subsets of W, i.e. a neighbourhood is every set $N \subseteq \mathscr{P}(W)$. ⊣

Definition 3.2.5. A neighbourhood N is called *non-emergent*, if for all sets $A, B \in N$ holds

$$\text{if } A \cup B \in N, \text{ then } A \in N \text{ or } B \in N$$ ⊣

Example 3.2.1. Let the set W be {1,2,3}. Then for example the neighbourhoods

- $\{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ and $\{\{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ are non-emergent

- $\{\{1\},\{2\},\{3\},\{2,3\},\{1,2,3\}\}$ and $\{\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ aren't non-emergent ⊣

This property can be tested according to the definition (brute-force approach). If we want to check if a neighbourhood N is non-emergent, we do the following:

For every set $C \in N$ consider all possible ways of splitting C into two sets A, B such that $A \cup B = C$ and then check the presence of at least one of them in N.

It's important to realize that whenever we want to test if a $C \in N$ satisfies the condition, we must consider *all* ways of splitting.

Example 3.2.2. The non-emergent cases in the previous example fail to satisfy the condition, because:

In neighbourhood $\{\{1\},\{2\},\{3\},\{2,3\},\{1,2,3\}\}$ the set $\{1,2,3\}$ can be split into

$\emptyset$	and	$\{1,2,3\}$;	OK, the latter is in N
$\{1\}$	and	$\{2,3\}$;	OK, both are in N
$\{2\}$	and	$\{1,3\}$;	OK, the former is in N
$\{3\}$	and	$\{1,2\}$;	OK, the former is in N
$\{1,2\}$	and	$\{2,3\}$;	OK, the latter is in N
$\{1,3\}$	and	$\{2,3\}$;	OK, the latter is in N
$\{1,2\}$	and	$\{1,3\}$;	NOT OK!, none is in N!

⊣

If we want to check all couples created by splitting a set W, we have to check $(3^{|W|}-1)/2$ couples.

Proposition 3.2.8. *Let Split be a function such that*

$$Split(W) = \{(A,B) \mid A, B \subseteq W;\ A \cup B = W\}.$$

Then $|Split(W)| = (3^{|W|}-1)/2$.

Proof. Split(W) works as follows:

$$\begin{array}{lcl} Disj\_split(W) & = & \{(A,B) \mid A \cup B = W; A \cap B = \emptyset\} \\ Extend((A,B),W) & = & \{(A,C) \mid B \subseteq C \subseteq W\} \\ Split(W) & = & \{(A,B) \mid (A,B) \in Extend((C,D)); (C,D) \in Disj\_split(W)\} \end{array}$$

Disj_split(W) splits into disjunctive couples. Then Extend((A,B),W) extends the second element of couple (A,B) in all possible way, such that the new couple still in union gives the whole set. Split(W) just collects extended disjunctive couples.

We demonstrate the algorithm on the set W={1,2}. We denote the sets $\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$ as 0,1,2 and 12, respectively.

$$
\begin{array}{lcl}
\text{Disj\_split(W)} & = & \{(0,12),(1,2),(2,1),(12,0)\} \\
\text{Extend((0,12))} & = & \{(0,12)\} \\
\text{Extend((1,2))} & = & \{(1,2),(1,12)\} \\
\text{Extend((2,1))} & = & \{(2,1),(2,12)\} \\
\text{Extend((12,0))} & = & \{(12,0),(12,1),(12,2),(12,12)\} \\
\text{Split(W)} & = & \{(0,12),(1,2),(1,12),(2,1),(2,12),(12,0),(12,1),(12,2),(12,12)\}
\end{array}
$$

Now, we will compute the size of these sets.

The size of Disj_split can be computed as follows. $|Disj\_split(W)| = \sum_{i=0}^{n} \binom{n}{i}$, where $n = |W|$, because in every step from $k = 0$ to n we choose a k-tuple from n and put it as a first item in the couple.

Then $|Split(W)| = \sum_{i=0}^{n} \binom{n}{i} 2^i$, because in every step we extend the second element by all possible subsets of the first element.

We prove, that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$ by induction on n. If $n = 0$, it works. Then for the induction case:

$$
\sum_{i=0}^{n+1} \binom{n+1}{i} 2^i = \sum_{i=0}^{n+1} (\binom{n}{i} + \binom{n}{i-1}) 2^i =
$$

$$
\sum_{i=0}^{n+1} \binom{n}{i} 2^i + \sum_{i=0}^{n+1} \binom{n}{i-1}) 2^i =
$$

$$
\sum_{i=0}^{n} \binom{n}{i} 2^i + \binom{n}{n+1} 2^{n+1} + \binom{n}{-1} 2^0 + \sum_{i=1}^{n+1} \binom{n}{i-1}) 2^i =
$$

$$
3^n + 2(\sum_{i=1}^{n+1} \binom{n}{i-1}) 2^{i-1} = 3^n + 2.3^n = 3^{n+1}
$$

Finally, we realize, that it the couples (A, B) and (B, A) are the same, so we will divide the result by 2. We also need to fine tune the result by subtracting one.

Hence, the size of $Split(W) = (3^{|W|} - 1)/2$. □

Thus, this aproach to testing is exponencial in the size of the basic set. We tried some other methods, all of them were exponencial.

We present one more method of testing the non-emergence of a neighbourhood. This method is based upon viewing a neighbourhood N as a graph $G = (V, E)$, where the vertices are elements of $\mathscr{P}(W)$.

We want to test if an element $M \in N$ satisfies the condition if $M = A \cup B$ then $A \in N$ or $B \in N$. We add an edge between A and B iff $A \cup B = M$. Now every vertex covering (a set $C \subseteq V$) represents a neighbourhood N', that satisfies the non-emergence for M.

If we add for every covered vertex the edges in the same way we did for M, we get a non-emergent cover.

Using all coverings we could generate all non-emergent neighbourhoods. Moreover, the minimal coverings would represent certain minimal non-emergent coverings, those which couldn't lose any of their sets without violating the condition.

The bad news is that the problem of finding the minimal vertex covering is is in general NP-complete. On the other hand, we don't necessarily need the minimal coverings and we consider coverings on a special type of graphs.

So, whether there is a method of testing non-emergence of a neighbourhood in polynomial time with respect to the size of the basic set – remains an open problem.

We conjecture it is a hard problem, as many other problems with set graphs.

3.2.4 Equivalent and converse formulas

In classical modal logics, there are generally much less formulas and inference rules than in normal modal logics.

In every classical modal logic, $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ is equivalent to formula $(\Diamond\phi \wedge \Diamond\psi) \to \Diamond(\phi \vee \psi)$. This formula is the dual of Distr$\vee$. We leave it to the reader as an exercise that every classical modal logic is closed to duals.

Again, the equivalent formula, $(\Diamond\phi \wedge \Diamond\psi) \to \Diamond(\phi \vee \psi)$, defines the class of non-emergent frames.

We call formula $(\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$ the *converse* to Distr$\vee$. A simple proof shows that this formula is provided by all normal modal logics. However, it is not contained by all classical logics, as can be shown by a neighbourhood countermodel. It can be shown that it defines following property on neighbourhood frames: if $A \in N(w)$ or $B \in N(w)$ then $A \cup B \in N(w)$. This can be proved in a similar style we used in section 3.2.1. This condition is by simple reasoning equivalent to: if $A \notin N(w)$, then all $B \subseteq A$, $B \notin N(w)$.

3.2.5 Applications

The property of non-emergence is hard to get grips on and we know of no domain that shows this type of behavior.

On the other hand, again, if we come to a phenomenon that we know can intuitively be described by formula $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$, we know that the domain must be non-emergent.

A note on the term "non-emergence". We chose this name, because it reflects the nature of the neighbourhood: if a set C is in the neighbourhood and can be split into two sets A, B which in union give C, then at least one of A, B must be in C. That means – C cannot simply emerge.

Chapter 4

Conclusions

4.1 Results and contribution

In this work we investigated the properties of the formula

$$\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$$

called *distribution of the modal operator over disjunction*, Distr$\vee$.

We considered this formula with respect to two types of modal logics, normal and non-normal.

In normal logics, we found a class of frames defined by this formula. It is the class of deterministic frames, i.e. frames (W, R) such that for all w, v_1, v_2 in W:

$$\text{if } Rwv_1 \text{ and } Rwv_2\text{, then } v_1 = v_2.$$

Then we showed soundness and strong completeness of logic KDistr$\vee$, the smallest normal logic providing KDistr$\vee$, with respect to the class of deterministic frames.

We managed to do this in spite of the fact that this formula is not in the form that automatically guarantees correspondence and strong completeness.

In non-normal logics we established a property of neighbourhood frames non-emergence. A neighbourhood frame (W, N) is *non-emergent*, iff for every world w and for all sets of worlds $A, B \subseteq W$:

$$\text{if } A \cup B \in N(w) \text{ then } A \in N(w) \text{ or } B \in N(w).$$

We proved, that Distr$\vee$ defines the class of non-emergent neighbourhood frames defined.

Then we explored the non-emergence; namely how to test frames for this property.

We also considered formulas equivalent to Distr$\vee$ and the formula representing the converse principle.

4.2 Further work

The (weak or strong) completeness of system $EDistr\vee$ with respect to the class of non-emergent neighbourhood frames remains an open problem.

The problem, whether there is a method of testing non-emergence of a neighbourhood in polynomial time with respect to the size of the basic set – remains open too.

In this work, we restricted ourselves to the basic modal language, i.e. language with only one unary modal operator. It would be interesting to look at how modalities of multi-modal languages interact with connectives.

A quite recent branch of modal logic is first-order modal logic, which is starting to be used in applications. Other direction of research is how extending to the first-order influences our results.

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Abstrakt

[Peter Drábik. Disjunkcia v Modlnej Logike. Diplomová práca. FMFI UK, 2007. 56 str.]

Diplomová práca sa skladá z dvoch častí.

Prvá časť práce podáva prehľad základných konceptov v modálnej logike: syntax a sémantika normálnych a non-normálnych modálnych logík, teória korektnosti a úplnosti a teória korešpondencie normálnych modálnych logík.

Druhá časť je venovaná skúmaniu distribúcie modálneho operátora vzhľadom na disjunkciu, teda vyšetrovaniu vlastností formuly $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$ v normálnych aj non-normálnych modálnych logikách. V práci charakterizujeme triedu rámcov definovanú touto formulou (tzv. trieda deterministických rámcov). Dokázaná je korektnosť a úplnosť najmenšej modálnej logiky obsahujúcej túto formulu. Triedu "neighbourhood"-rámcov definovaných touto formulou sa podarilo charakterizovať vlastnosťou non-emergencie. V práci je skúmané aj testovanie non-emergencie "neighbourhood"-rámcov.

Kľúčové slová: modálna logika, non-normálna, disjunkcia, distribúcia, neighbourhood, non-emergencia