Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Edge colouring of signed cubic graphs

Master's Thesis

2024
Bc. Bohdan Jóža

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Study Programme: Computer Science
Field of Study: Computer Science
Department: Department of Computer Science
Supervisor: doc. RNDr. Robert Lukoťka, PhD.

Bratislava, 2024
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Názov: Edge colourings of signed cubic graphs
Hranové farbenia signovaných kubických grafov

Anotácia: Signované grafy sú grafy, ktorých hrany sú ohodnotené prvkami z {-1, 1}. Prepínanie signovaného grafu v jeho vrchole v je vynásobenie ohodnotenia incidentných hrán hodnotou -1. Grafy, ktoré možno získať sériou operácií prepínania sú ekvivalentné. Existuje veľa článkov, ktoré skúmajú rozšírenie štandardných grafových invariantov na signované grafy. Jednou zo skúmaných tém je farbenie signovaných grafov. Predmetom práce budú hranové farbenia signovaných kubických grafov. Hranové farbenia signovaných grafov začal skúmať Behr v článku [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. Cieľom práce je začať systematické štúdium hranovej 3-zafarbiteľnosti signovaných grafov.

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Spôsob sprístupnenia elektronickej verzie práce:
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Dátum zadania: 16.11.2022

Dátum schválenia: 28.04.2023

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THESIS ASSIGNMENT

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Title: Edge colourings of signed cubic graphs

Annotation: Signed graphs are graphs, whose edges have assigned values from {-1, 1}. Switching at a vertex v of a graph is done by multiplying the values of all edges incident with v by -1. Graphs that can be obtained from each other by switching are called equivalent. There are plenty of papers studying generalization of standard graph invariants to signed graphs. One of these invariants is graph colouring. The thesis should focus on edge colourings of signed cubic graphs. The study of edge colourings of signed graphs was started by Behr [Edge coloring signed graphs, Discrete Mathematics 343(2020)]. The aim of the thesis is to initiate the systematic study of 3-edge-colourability of signed cubic graphs.

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Abstrakt

Signované grafy boli vynájdené v roku 1953 vďaka Frankovi Hararymu ako model na štúdium vzťahov ale problém farbenia v tejto oblasti nebol preskúmaný do roku 1982, kedy Thomas Zaslavsky zverejnil prvé výsledky. Veľa základných kameňov teórie grafov však bolo do signovaných grafov premostených len nedávno. V tejto práci prezentujeme algoritmus na generovanie neekvivalentných signovaných grafov, ktoré nie sú hranovo 3-zafarbiteľné spolu s vybranými výsledkami a prvotnou analýzou.

Kľúčové slová: signovaný graf, kubický graf, hranové farbenie, snark, generovanie neekvivalentných grafov

Abstract

Signed graphs were invented by Frank Harary in 1953 as a model for studying social networks but the problem of coloring was not explored until 1982 when Thomas Zaslavsky published his first results. However, much of the graph theory fundamentals was not established until recently. In this thesis we continue the research of 3-edge-colorability of signed graphs. We present an algorithm that generates non-equivalent signed graphs that are not 3-edge-colorable along with its results and preliminary analysis.

Keywords: signed graph, cubic graph, edge coloring, snark, generating non-equivalent graphs

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Introduction

Despite the fact that the problem of graph colouring has been studied for a long time and is still being relentlessly studied today, there are still areas to explore. Edge colouring is to this day a novel topic in combination with the concept of signed graphs.

First discovered by the mathematician Frank Harary in 1953 as a model for studying social networks, signed graphs colouring remained an idle topic until 1982 when Thomas Zaslavsky published multiple seminary papers on the topic. Many fundamental results in the study of nowhere-zero flows and the chromatic number of signed graphs have been established only recently and research the problem of edge colouring was started by Richard Behr in 2020. The goal of this thesis is to start a systematic study of 3-edge-colourability of signed cubic graphs.

In the first chapter we will define key concepts in the signed graph theory and offer an overview of the research done so far in this topic. We also describe the relationship to unsigned graphs and how it results in the desirable properties of the color set. In the second chapter we outline how systematic generation of signed graphs that are not 3-edge-colorable is achieved. Details regarding the implementation are also provided. In the third chapter we present our results so far, which include trivial requirements for edge-colorability. Finally, we provide options for future research that can be pursued.

Chapter 1

Preliminary Graph Theory

First, let's define some basic concepts of graph theory, starting with the graph itself.

1.1 Graphs

A graph is an algebraic structure most commonly used to describe relationships between objects. There are many definitions of a graph. The most abstract definition of a graph is simply a set V and a relation R on V denoting which elements of V are connected. Graphs in general are *directed*; if R is symmetric, the graph is *undirected*. For the purposes of this work we will be using a geometric definition and generally undirected graphs.

Definition 1.1. An undirected graph is an ordered pair $G = (V, E)$, where V is a set of *vertices* and E is a set of edges, i. e. a set of unordered pairs of vertices $\forall e \in E: \; e = (u, v); u, v \in V$.

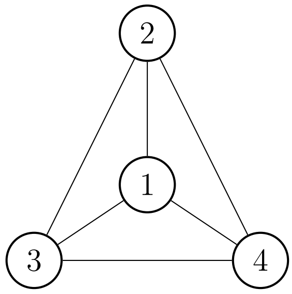
Definition 1.2. A *path* in a graph G from v to w; $v, w \in V$ is a sequence of vertices $(u_1, u_2, \ldots, u_n)$; $\{u_i \mid 1 \leq i \leq n\} \subseteq V$ such that $u_1 = v$, $u_n = w$ and $\{(u_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \subseteq E$. A graph is *connected* if there exists a path between every pair of vertices $v, w \in V$; $v \neq w$.

Definition 1.3. A *degree* $\Delta(v)$ of a vertex v denotes how many edges are incident to this vertex. $\Delta(G)$ is the highest degree of any vertex in G.

Definition 1.4. A graph is *k-regular* if the degree of each vertex is exactly k. A *cubic graph* is a 3-regular graph.

As an example, the K_4 graph is cubic.

In general statements about graphs in later chapters, we are referring to unordered cubic graphs.



1.1.1 Colouring

When simple binary relationships between objects are not enough, weighted graphs and colouring offer a wider range of applications. Assigning colours to vertices or edges of graphs makes classifications of these objects possible.

Definition 1.5. A vertex colouring of a graph G is a mapping from the vertex set of G to a set of colours C. An edge colouring of a graph G is a mapping from the edge set of G to a set of colours C.

Definition 1.6. A *proper vertex colouring* of G is a vertex colouring such that no two neighboring vertices share a colour. A *proper edge colouring* is an edge colouring such that no two edges that share an endpoint have the same colour. A proper colouring using k colours is called a *k-colouring*.

As colouring in general is not very interesting, we will be considering only proper colourings henceforth. It is also important to define the set of "colours", especially when colouring signed graphs. Although actual colours tend to be a nice visualization of a colouring, it is more practical to use a subset of integers $C \subseteq \mathbb{Z}$.

The canonical colouring problem is to find the minimum number of colours required for a proper colouring. This number is called the *chromatic number* for vertex colourings and *chromatic index* for edge colourings. Determining the chromatic number and index is useful in other areas of graph theory as well.

Theorem 1.1. *A graph is bipartite if and only if it has a proper vertex 2-colouring.*

For regular unsigned graphs these numbers are known.

Theorem 1.2 (Brooks[1])**.** *The chromatic number of a graph G is $\Delta(G)$ for all graphs except complete graphs and cycles of odd length, where the chromatic number is $\Delta(G)+1$.*

Theorem 1.3 (Vizing)**.** *The chromatic index of a simple graph G is $\Delta(G)$ or $\Delta(G)+1$.*

In other words, we can always colour the edges of a graph using at most $\Delta(G)+1$ colours where $\Delta(G)$ is the highest degree of any vertex in G. The lower bound $\Delta(G)$ is trivial; we need exactly $\Delta(G)$ colours at the highest degree vertex in G to construct a proper colouring. The Vizing theorem proves the upper bound using Kempe chains.

1.2 Signed graphs

A signed graph is a graph in which each edge has either a positive or a negative sign. There are multiple definitions of a signed graph but for our purposes a sign function is most practical.

Definition 1.7. A *signed graph* $\Gamma = (G, \sigma)$ consists of a *underlying graph* G and a *sign function* $\sigma : E(G) \to \{+, -\}$ that assigns a sign to each edge of G.

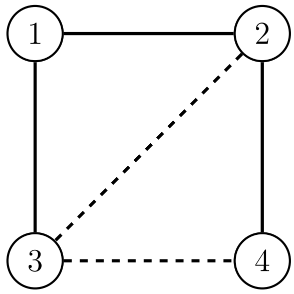


Figure 1.1: Example of a signed graph. Dashed lines indicate negative edges, solid lines positive edges.

A fundamental concept in the signed graphs theory is *balance.* The sign of a path is the product of the signs of its edges. A path is positive if and only if there is an even number of negative edges on it. A cycle is balanced if it is positive and a signed graph is balanced if each cycle in it is balanced[2].

Theorem 1.4 (Harary)**.** *A signed graph is balanced if and only if*

1. *for every pair of vertices, all paths between these vertices have the same sign*
2. *the vertices can be divided into two subsets (possibly empty) such that each edge with both ends in the same subset is positive and each edge with ends in different subsets is negative*

This is a generalization of the earlier mentioned bipartite graph theorem (Theorem 1.1).

The proof uses the method of *switching.* Switching a vertex of a signed graph reverses the sign of each edge incident to it. More generally, switching a signed graph reverses the sign of each edge between a vertex subset and its complement.

We can prove by induction that a signed graph can be switched to an all-positive graph if and only if it is balanced. Both conditions in Harary's theorem apply to all all-positive graphs and graphs that can be switched from an all-positive graph. Consequently, all balanced graphs are equivalent to an all-positive graph, which is an alternative definition of a positive graph. Similarly, we call a graph *antibalanced* if it is equivalent to an all-negative graph, (all cycles of even length in and antibalanced graph are positive and cycles of odd length are negative).

Definition 1.8. If a signed graph can be obtained from another signed graph by switching, they are considered *equivalent*. For a single underlying graph, switching forms *equivalence classes* of signed graphs. Within a single equivalence class all graphs can be switched to each other.

It makes sense to study properties of signed graphs that behave consistently under switching. An example of such property is the sign of cycles. Switching a single vertex doesn't change the sign of cycles (cycles containing the vertex reverse signs for two edges resulting in the same product) and switching a set of vertices is equivalent to a sequence of one-vertex-switches (each edge within the set and within the complement gets reversed twice).

1.2.1 Colouring

The research in signed graph colouring was initiated by Zaslavsky[3] in the early 1980s and published in multiple seminal papers[4, 5, 6].

Definition 1.9. A *signed vertex colouring* $\phi(\Gamma)$ of a graph Γ is a mapping from the vertex set of Γ to a set of signed colours C. A *signed edge colouring* $\gamma(\Gamma)$ of a graph Γ is a mapping from the set of half-edges (vertex-edge incidences) of Γ to a set of colours C. Additionally, the half-edges must have the same colour on positive edges and opposite colours on negative edges.

$$(\forall e = (u, v) \in E(\Gamma)) \quad \gamma(e, u) = \sigma(e)\gamma(e, v)$$

Definition 1.10. A *proper vertex signed colouring* is a colouring $\phi(\Gamma)$ such that for each pair of neighboring vertices (u, v) $\phi(u) \neq \sigma(uv)\phi(v)$. In case of *proper edge signed colouring* the definition remains the same, because the colouring condition is already a part of the general colouring definition. Each colour must be present at each vertex at most once (or adjacent half-edges have different colours). We are, again, assuming only proper colourings from now on.

Here it is even more important to define the colour set. Zaslavsky[5] defined a k-colouring based on a signed colour set $C_k = \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$ and called colourings zero-free if the colour 0 was not used. He then studied the properties of *chromatic polynomials* related to signed colourings, the number of colourings for a signed graph. (Balanced chromatic polynomials in case of zero-free colourings.)

This definition is consistent under switching. Assuming a graph Γ and a proper vertex colouring ϕ, if we obtain Γ' by switching vertex u, then $\phi' = \phi; \phi'(u) = -\phi(u)$ is a proper vertex colouring of Γ'. Similarly for edge colouring in which we reverse the sign of each half-edge incident to u.

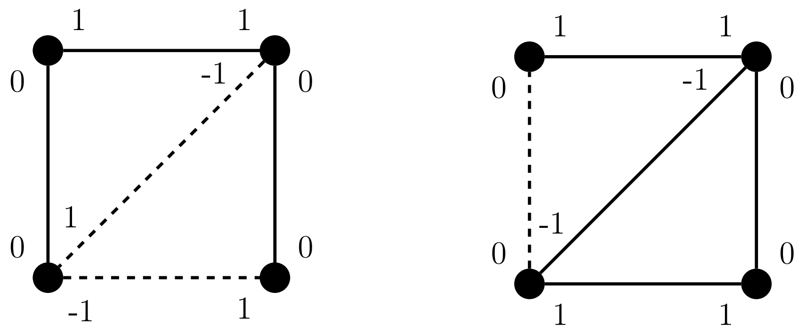


Figure 1.2: Example of a proper signed edge colouring on the left. We obtain the graph on the right by switching the bottom left vertex and the colouring remains correct and proper.

However, this definition is not a natural extension of the original colour set of integers, because a k-colouring essentially uses $2k$ or $2k+1$ signed colours. It is a desirable property for the colour set because signed graphs themselves are an extension of unsigned graphs. A balanced signed graph is essentially equivalent to the unsigned underlying graph, so its chromatic number and index for instance should also match. In *The chromatic number of a signed graph*, Máčajová et al. define the colour set a little bit differently: An n-colouring uses the colour set $C_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$ and $C_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k+1$. We adopt this colour set in this thesis.

We adopt the signed versions of colouring definitions from *The chromatic number of a signed graph*[7] and *Edge colouring signed graphs*[8]. In the latter, however, Behr colours the half-edges the same colour if their sign is negative, not positive. Since there was no obvious advantage stated in the article and the definitions are somewhat equivalent, we find this version to be more natural, as the underlying graph can be interpreted as an "absolute value" of the signed graph, which is positive.

1.3 Motivation

"In the study of various important and difficult problems in graph theory (such as the cycle double cover conjecture and the 5-flow conjecture), one encounters an interesting but somewhat mysterious variety of graphs called snarks. In spite of their simple definition [...] and over a century long investigation, their properties and structure are largely unknown." — Chladný, Škoviera [9]

By Vizing's theorem, cubic graphs are colourable either with three ("class one" graphs) or four colours ("class two") graphs. The exact definition of a snark may vary from paper to paper but a snark is essentially a cubic graph with chromatic index four (its edges can't be coloured with three colours). The definition varies across papers, trivial cases are generally excluded. Every cubic graph with a loop or a bridge is a "snark", triangles (cycles of length three) can be contracted into a single vertex and cycles of length four can also be simplified. Therefore many definitions forbid these properties by considering true snarks only graphs with girth (length of the shortest cycle) at least five. Even more strongly, only cyclically 4-edge-connected graphs are considered (there is no subset of three or fewer edges such that their removal will disconnect the graph into two subgraphs each containing a cycle). One of the alternative formulations of the four colour theorem is that each snark is non-planar. Snarks are important in a multitude of graph theory areas and thus it makes sense to investigate the reach of signed snarks too.

1.4 Previous research

In *The chromatic number of a signed graph*[7] Máčajová et al. continue Zaslavsky's research by studying the properties of the chromatic number of signed graphs, ultimately proving a signed version of the famous Brooks'[1] theorem.

Theorem 1.5 (Signed Brooks' Theorem). *Let* Γ *be a simple connected signed graph. If* Γ *is not a balanced complete graph, a balanced odd circuit or an unbalanced even circuit, then* $\chi(\Gamma) \leq \Delta(\Gamma)$.

Edge colouring signed graphs defines a version of the signed edge colouring and proves a signed version of the equally fundamental Vizing's theorem.

Theorem 1.6 (Signed Vizing's Theorem). *Let* Γ *be a simple signed graph. The chromatic index of* Γ *is* $\Delta(\Gamma)$ *or* $\Delta(\Gamma) + 1$.

Chapter 2

Generating signed snarks

Since the structure of snarks is generally unknown, the most efficient way of systematically generating snarks is still a brute-force approach.

2.1 Chromatic index problem

To determine the chromatic index of a cubic graph is an NP-complete problem. By extension, determining the chromatic index of a signed cubic graph is also NP-complete, because of the trivial reduction from signed chromatic index problem to unsigned chromatic index problem. Instead of designing an algorithm we decided to implement a conversion from the chromatic index problem to 3SAT and using a highly optimized SAT solver anticipating better effectiveness.

2.1.1 Conversion to 3SAT

For any cubic signed graph Γ we will construct a 3SAT formula $F(\Gamma)$ that is satisfiable if and only if the graph is 3-colourable. There will be three literals for each half-edge ev of Γ, one for each colour from $C_3 = \{-1, 0, 1\}$. Let these be x_{ev}^{-1}, x_{ev}^{0} and x_{ev}^{1}. In any evaluation of these literals that satisfy F exactly one of them will be true denoting the colour of the half-edge. This will be guaranteed using three constituent formulas. Let $\Gamma = ((V, E), \sigma)$

$$F_1 = \bigwedge_{e=vw \in E} (x_{ev}^{-1} \vee x_{ev}^{0} \vee x_{ev}^{1}) \wedge (x_{ew}^{-1} \vee x_{ew}^{0} \vee x_{ew}^{1})$$

The first formula ensures that each half-edge is coloured and is the only set containing clauses of length 3. The next formula will enforce the correctness of the colouring, restricting the colours of half edges that for one complete edge. Illegal signatures for each edge are negated using DeMorgan rules, resulting in a convenient CNF form. No edge can be coloured 0 on one side and 1 or -1 on the other $(\neg(x_{ev}^{0} \wedge x_{ew}^{1}) = (\neg x_{ev}^{0} \vee \neg x_{ew}^{1}))$

and the colours must be the same if the edge is positive $((\neg x^1_{ev} \vee \neg x^{-1}_{ew}))$ and opposite if the edge is negative $((\neg x^1_{ev} \vee \neg x^1_{ew}))$.

$$F_2 = \bigwedge_{e=vw \in E} (\neg x^0_{ev} \vee \neg x^1_{ew}) \wedge (\neg x^0_{ev} \vee \neg x^{-1}_{ew}) \wedge (\neg x^{-1}_{ev} \vee \neg x^{\sigma(e,w)}_{ew}) \wedge (\neg x^1_{ev} \vee \neg x^{-\sigma(e,w)}_{ew}) \wedge (\dots v \rightleftarrows w \dots)$$

The first four clauses illustrate the condition from the "perspective" of v, they will be repeated for w as well by switching instances of v and w. Lastly we need to ensure the colouring is proper. Let $N(v) = \{(v,w) \mid (v,w) \in E;\ w \in V\}$ be the set of edges incident to v.

$$F_3 = \bigwedge_{\substack{v \in V \\ e_1, e_2 \in N(v);\ e_1 \neq e_2}} (\neg x^{-1}_{e_1 v} \vee \neg x^{-1}_{e_2 v}) \wedge (\neg x^0_{e_1 v} \vee \neg x^0_{e_2 v}) \wedge (\neg x^1_{e_1 v} \vee \neg x^1_{e_2 v})$$

Each pair of half-edges with a common vertex has to have different colours. Note that we don't need to explicitly ensure that for each half-edge exactly one literal is true, only that at least one is true, because it is a consequence of the properness of the colouring.

Theorem 2.1. *3SAT formula $F(\Gamma) = F_1 \wedge F_2 \wedge F_3$ constructed in the way described above is satisfiable if and only if Γ is 3-colourable.*

Proof. Follows from the construction of F encapsulating all properties of a proper signed 3-colouring. □

2.2 Equivalence

Signed graphs can be equivalent in a combination two ways, switching-equivalent or isomorphic. Let's explore the switching equivalence first.

2.2.1 Signed equivalence classes

On any base graph G there are $2^{|E(G)|}$ possible signed graphs. Zaslavsky[3] enumerated the switching equivalence classes and described a representative for each class.

Theorem 2.2. *Let G be a simple unsigned base graph and $T \subseteq E(G)$ a spanning tree of G. Then all signed graphs that have an all-positive signature on T are not switching-equivalent and each equivalence class based on G has exactly one representative among them. There are $2^{|E(G)|-|V(G)|+1}$ switching classes on G.*

Proof. Take any signed graph constructed this way. Switching no vertices and all vertices results in the same graph. To obtain a different graph, at least one vertex will not be switched and at least one vertex will be switched. The set of switched vertices $A \neq \emptyset$ and the set of untouched vertices $B \neq \emptyset$ are a partition of $V(G)$. Since G is connected, there is at least one edge between A and B and at least one of them is in T. This edge will change its sign based on the definition of switching. So any graph we obtain by switching one of the graphs from theorem 2.2 will not be all-positive on T, making all these graphs belong to different equivalence classes. □

According to Theorem 2.2, on a base cubic graph with n vertices there are $2^{\frac{n}{2}+1}$ equivalence classes, one for each signature of edges that are not in the spanning tree. Note that $\frac{n}{2}$ is always an integer since the number of vertices in a cubic graph is even. The following algorithm generates all non-equivalent representatives.

2.2.2 Generating algorithm

The algorithm first finds a spanning tree and assigns positive signs to all edges in it. Edges are enumerated and the spanning tree edges will be ignored. We can now imagine that positive sign means zero and negative sign means one. The remaining edges form a binary number in this way. To obtain the next representative we simply increment this number by one. This means flipping the lowest consecutive sequence of ones and the first instance of zero. We keep reversing the sign of edges from lowest to highest until we flip a positive edge for the first time or run out of edges. If we run out of edges, we basically went from the number $2^{\frac{n}{2}+1} - 1$ to 0. So starting with any signature that is all-positive on the spanning tree, we will have generated all equivalence classes after $2^{\frac{n}{2}+1}$ incrementations. The spanning tree, however, has to remain the same during the entire process.

2.2.3 Isomorphism

Signed graphs can be isomorphic if and only if their base graphs are isomorphic. We get the homomorphism by taking away signatures. This is a key observation because if we base our signed graphs on non-isomorphic graphs the only candidates for isomorphisms are based on the same graph. In Enumerating Switching Isomorphism Classes of Signed Graphs[10] non-equivalent graphs are enumerated using a one-to-one correspondence between switching isomorphism classes and signed double covers of Γ. The algorithm first generates all possible signed graphs from a base graph and then filters them for isomorphisms. This approach, although correct, is too inefficient for our purposes. The algorithm generates all signed graphs for each base graph, which for $n = 18$ is already too much, 41301 cubic graphs with 18 vertices results in 10.8 billion signed graphs.

A second approach is based on the cycle space of Γ and Eulerian graphs.

Theorem 2.3 (Zaslavsky, Cameron[10]). *There is a one-to-one correspondence between switching isomorphism classes of* Γ *and Aut(G)-isomorphism classes of Eulerian subgraphs of* G.

This approach results in a faster enumeration algorithm, but it does not provide means to generate non-isomorphic signed graphs. This is a point of possible future research as there might be a way to utilize this theory in a faster generation algorithm.

In this thesis we are using a simple approach for filtering signed graphs for isomorphisms. The signed graphs are converted to unsigned graphs differentiating negative edges by inserting a vertex in the middle. Now we can use known filtering algorithms, finding the canonical form for each new graph and comparing it against already seen graphs.

Definition 2.1. A *canonical form* is a labeled graph Canon(G) that is isomorphic to G such that every graph isomorphic to G has the same canonical form. To compute whether graphs G and H are isomorphic we compute their canonical forms and test whether they are identical.

Finding the canonical form is as hard as determining whether two graphs are isomorphic and as of today it is unknown if a polynomial deterministic algorithm exists that solves this problem. However, here we are working with small graphs (less than 26 vertices) and special cases of bigger graphs, so time costs related to the number of vertices are more or less irrelevant. What is relevant is that this approach has quadratic complexity with regards to the number of signed graphs on each base graph as we are comparing each new canonical form to potentially many previous ones. This cost is mitigated somewhat by filtering the graphs for signed snarks first and for isomorphisms second.

2.3 Implementation

We achieved our results using the following implementation. The programming language of choice was C++ over Python due to its speed and a base of tools for graph computation. We implement a simple data structure to represent signed graphs as opposed to nauty and other optimized structures because there is little support for signed graphs "out of the box". Additionally, there is no need to optimize for the graph size. Cubic graphs and unsigned snarks are generated using snarkhunter. Our SAT solver of choice is the winner of the SAT Competition 2020[11], kissat. It is a “condensed and improved reimplementation of CaDiCaL in C”.

Chapter 3

Results

We found all signed snarks up to 18 vertices.

N	G	non-equivalent signatures per G	signed G	signed snarks
4	1	8	8	0
6	2	16	32	0
8	5	32	160	1
10	19	64	1216	48
12	85	128	10 880	227
14	509	256	130 304	2768
16	4060	512	2 078 720	31 869
18	41 301	1024	42 292 224	437 381

Figure 3.1: Basic signed graph data. Here signed snarks were not yet filtered for isomorphisms.

The smallest signed snark is smaller than the Petersen graph (smallest snark), it is the projection of a cube.

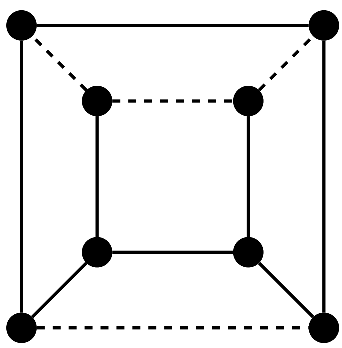


Figure 3.2: Smallest snark

Similarly to regular snarks, there are trivial properties of signed graphs that don't allow the possibility of a 3-edge-colouring. The following unsigned graph is the smallest graph that doesn't have a 3-edge-colourable signature.

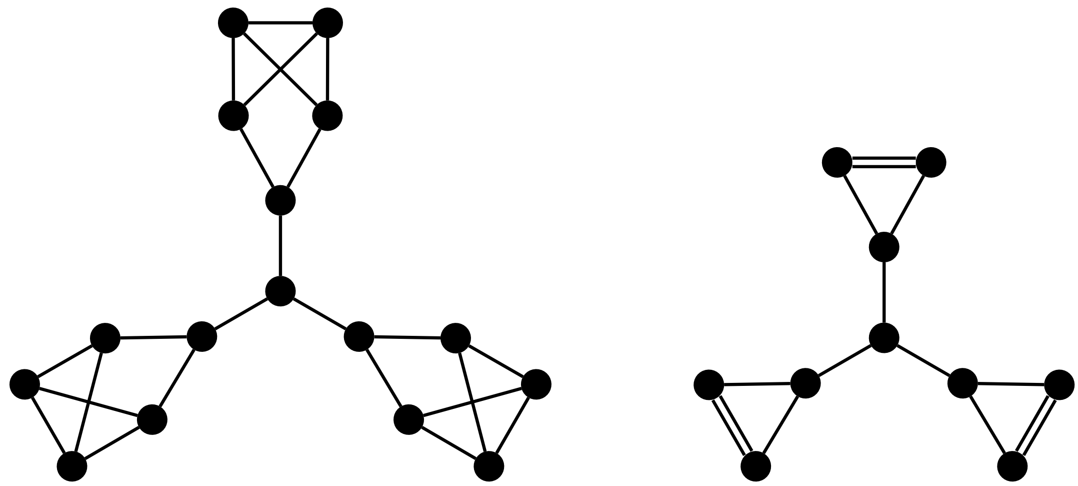


Figure 3.3: Smallest simple graph without a 3-edge-colourable signature and a simplified version allowing duplicate edges.

Theorem 3.1. *An unsigned graph G has a signature that admits a 3-edge-colouring if and only if it has a 1-factor (1-regular subgraph with the same vertex set).*

Proof. If there is a signature and a 3-edge-colouring on it, the edges coloured 0 form by the definition of a proper edge colouring a perfect matching. Now let $M \subseteq E(G)$ be a 1-factor. Let's assign the colour 0 to these edges again and remove them from G. After removing a 1-factor from a cubic graph we obtain a 2-factor, a set of disjunct cycles (if two cycles would have a common vertex, its degree in the original graph would have to be at least 4). According to Theorem 1.5 for the cycles to be colourable, we assign any balanced signature to even cycles and any unbalanced signature to odd cycles. All cycles from this 2-factor will now be 2-edge-colourable with colours 1 and -1 and combined with the 0-coloured 1-factor we obtain a 3-edge-colourable signed cubic graph. $\square$

The graph in Chapter 3 has no 1-factor. (The middle vertex has to be connected to one of the three triangles and the other two triangles will not have a matching.)

3.0.1 Future research

There are multiple directions we intend to take our research into this topic in the future. The analysis of small signed snarks can be taken further by inspecting different classes of graphs and searching for similarities. By optimizing the filtering algorithm, bigger graphs can be included.

Conclusion

In this thesis we outlined an algorithm to filter signed graphs that are not 3-edge-colorable. We analysed the first results and showed that for any 3-edge-colorable signature a cubic graph has to admit a perfect matching.

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Appendix A

Source code

The latest version of the source code can be found on `https://github.com/Bohdanator/signed-cubic-graphs`