Circular chromatic index of small snarks

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Introduction

Suppose $G = (V, E)$ is a graph.

- A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices
- An edge coloring of a graph is an assignment of labels or colors to each edge such that no two edges that are incident with the same vertex are identically colored.
- Line graph of graph G is denoted as $L(G)$
  - $V(L(G)) = E(G)$
  - $\forall e, e' \in E(G)$ are adjacent in $L(G)$, if they are incident with common vertex in G.

Circular Coloring

Suppose $G = (V, E)$ is a graph and C a circle of (euclidean) length r.

- r-circular coloring c of G
  - c assigns each vertex x an open unit length arc $c(x)$
  - $\forall e = (x, y) \in E(G) : c(x) \cap c(y) = \emptyset$
- circular chromatic number
  $\chi_c(G) = \inf\{r : G \text{ is } r\text{-circular colorable.}\}$

Interval Coloring

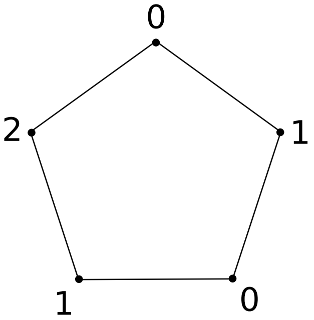
Suppose $G = (V, E)$ is a graph and C a circle of (euclidean) length r.

- r-interval coloring g of G
  - g assigns each vertex x an open unit length sub-interval $[0, r]$
  - $\forall e = (x, y) \in E(G) : g(x) \cap g(y) = \emptyset$
- chromatic number
  $\chi(G) = \min\{r : G \text{ there is an } r\text{-interval coloring.}\}$

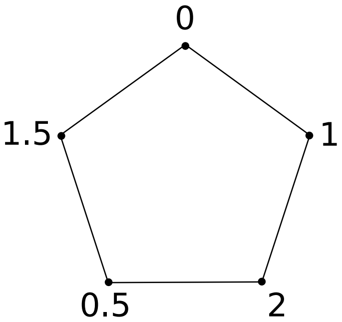
Theorem ([Zhu01, Zhu06])

For any finite graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.

Interval vs. Circular coloring example



$\chi(G)=3$



$\chi_c(G)=2.5$

Circular chromatic index

Suppose $G = (V, E)$ is a graph.

- chromatic index $\chi'(G) = \chi(L(G))$
- circular chromatic index $\chi'_c(G) = \chi_c(L(G))$

Theorem

For any finite graph G, $\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G)$.

Motivation

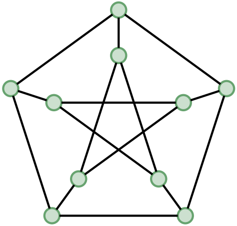
- By Vizing's theorem, chromatic index of cubic graphs is either three or four.
- χ'_c is measure of uncolorability
- NP-complete problem
- number of colors is proportional to number of edges (guesstimate is $\#\ \text{colors}^{\#\ \text{edges}}$ resp. $\#\ \text{edges}^{\#\ \text{edges}}$)

Snark Graphs (1)

Definition

Snark – connected bridgeless cubic graph with chromatic index four.

- Avoiding trivial cases
  - cyclically 4-edge connected
  - girth ≥ 5
- Petersen graph is the smallest snark (of order 10)



Snark Graphs (2)

n	# snarks
10	1
18	2
20	6
22	20
24	38
26	280
28	2900
30	28399
32	293059
34	3833587
36	60167732

Table 1: Database House of graphs [Hou] in graph6 format [McK]

Potential indices (1)

- All rational numbers $\frac{p}{q}$ between 3 and 4, such that $p \leq \mathit{circumference}(L(G)) \leq |E(G)|$
- ordering of s potential indices
  - ascending
  - ascending according to numerator

Potential indices (2)

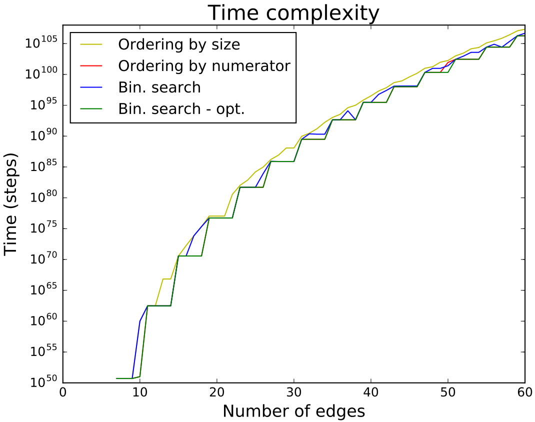


Figure 1: Time complexity of determining circular colorability

Backtrack - basic algorithm

- Ordering of edges – next edge is edge with most colored neighbours
- First edge has color 0
- Second edge has half of colors
- Try only colors that are not covered by edge neighbours

SAT solvers [Lin]

For graph G and coloring c ($\frac{p}{q} - \mathit{coloring}$) we represent this problem as SAT instance in CNF

- $\forall e \in E(G)$ let $P_{e,y} \Leftrightarrow c(e) = y$.
- clause type 1: $\forall e \in E(G)$ – $\bigvee P_{e,y} \qquad y \in \{0..(p-1)\}$
- clause type 2: $\forall e, f \in E(G)$ (e and f are incident) – $\neg P_{e,y} \vee \neg P_{f,z} \qquad |y-z| < q \vee |y-z| > p-q$

SAT solvers [Lin] – alternative representation (1)

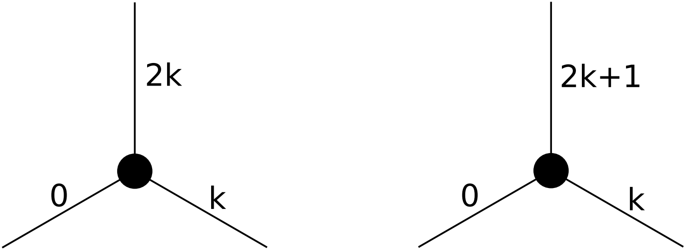


Figure 2: Colors in node for (3k+1, k)-coloring

SAT solvers [Lin] – alternative representation (2)

- Variables – $\forall v \in V(G)$ let $P_{v,t} \Leftrightarrow c(v) = t$
- Clause type 1 –
  $\forall v \in V(G) \ \bigvee P_{v,t} \qquad t \in \{\text{all possible triplets of colors}\}$
- Clause type 2 – $\forall u, v \in V(G) \wedge (u, v) \in E(G)$ – $\neg P_{u,t_1} \vee \neg P_{v,t_2}$ if not he same color is assigned to the edge (u, v) by triplets t_1 and t_2.

Theoretical time complexity

Algorithm	Complexity
Exhaustive search	$O(p^m)$
Backtrack	$O\left(\sqrt(2)^{m-3}\left(\frac{p}{3}\right)^{m-1}\right)$
SAT instance edges	$O\left(\left(1.32^{p\left(2-\frac{3}{p}-\frac{19}{6m}+\frac{5}{mp}\right)}\right)^m\right)$
SAT instance vertices	$O\left(\left(1.32^{p\left(\frac{2}{3}(q^2+q)+5\right)-3}\right)^m\right)$

Table 2: Time complexity of method determining (p, q)-circular edge colorability of a given graph

Algorithm comparison – finding index

Order	# snarks	Backtrack	SAT solver [Lin]
10	1	0m0.02s	0m0.09s
18	2	$>$10h	0m5.97s
20	6		0m47.08s
22	20		3m26.19s
24	38		8m27.99s
26	280		33m39.60s
28	2900		49h26m27.67s
30	28399		*204h22m42.82s

Table 3: Backtrack and SAT solver comparison.

* 7 threads were used to compute results

Time vs. order of graph

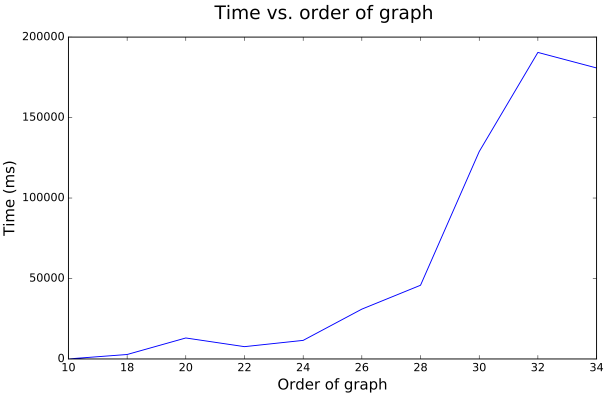


Figure 3: Average running time for snark depending on a given order. Results were computed by SAT solver.

Graphs with $(10, 3)$ circular chromatic index

- Generate all $\frac{10}{3}$-colorings
- Determine if for each coloring c, $D_c(G)$ contains a tight cycle
- Generating colorings
  - Backtrack
  - All solution sat

Algorithm comparison – generating all colorings + testing tight cycles

Order	# snarks	$(10,3)$-index	Backtrack	All solution sat [CNF16]
10	1	0	0m0.05s	0m0.21s
18	2	1	0m11.68s	0m7.21s
20	6	5	0m33.81s	0m11.68s
22	20	18	8m14.48s	1m48.01s
24	38	37	47m11.87s	9m47.51s
26	280	211		5h31m7.53s

Table 4: $(10,3)$-index of graphs of order less than 28

Results

Order		10	18	20	22	24	26	28	30
#snarks / Index		1	2	6	20	38	280	2900	28399
(29, 9)	$3.\overline{22}$	0	0	0	0	0	**1**	0	**8**
(13, 4)	3.25	0	0	0	0	0	**13**	**314**	**4130**
(23, 7)	3.29	0	0	0	0	**1**	**55**	**1076**	**12775**
(33, 10)	3.30	0	0	0	0	0	0	0	**1**
(10, 3)	$3.\overline{33}$	0	**1**	**5**	**18**	**37**	**211**	**1509**	**11483**
(17, 5)	3.40	0	0	**1**	**2**	0	0	**1**	**2**
(7, 2)	3.50	0	**1**	0	0	0	0	0	0
(11, 3)	$3.\overline{66}$	**1**	0	0	0	0	0	0	0

Table 5: Results for graphs of order up to 30

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Thank You for Attention

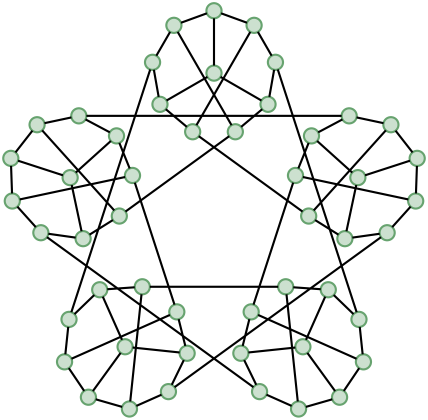

Figure 4: The Szekeres snark

Question (1)

Algoritmus na určenie poradia čísel r vychádzal z predpokladu, že cirkulárne chromatické index sú distribuované rovnomerne.
Z výpočtov vyplýva, že väčšina snarkov má index 13/4, 23/7 alebo 10/3. Mohli by ste navrhnúť a vyskúšať stratégiu založenú na tomto predpoklade?

- Usporiadanie zlomkov podľa frekvencie – [10/3, 23/7, 13/4]
- Na prehlásenie, že daný zlomok je index je nutné overiť aj zvyšné potenciálne zlomky – zamietnuť menšie alebo vygenerovať všetky farbenia a preveriť tesné cykly

Question (1)

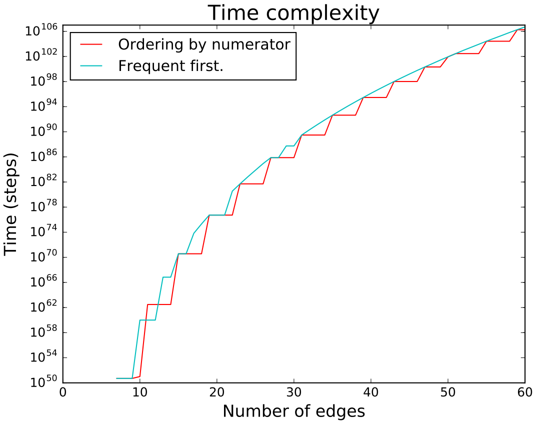

Figure 5

Question (2)

Okrem obmedzenia na farby prvých dvoch hrán nie je popísaná žiadna snaha o vylepšenie (odhadu) zložitosti algoritmu spätného prehľadávania.

- V práci je na stranách 23-26 opísaná vylepšená verzia backtrack algoritmu s fixným poradím hrán, optimalizujúca počet skúšaných farieb. My sme vychádzali v práci priamo z tohto algoritmu.

Question (3)

Jedno z možných vylepšení je nefarbiť (zafarbiteľné) hrany so štyrmi zafarbenými susedmi, čo dáva horný odhad jednej farby.

- Analýza možných prípadov

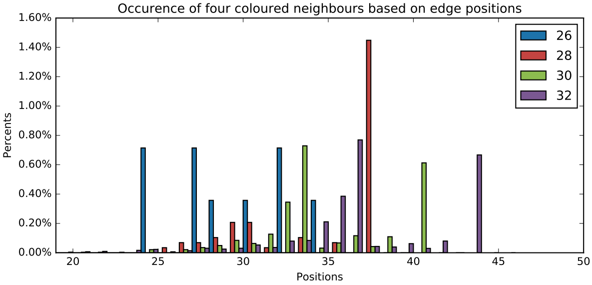

Figure 6

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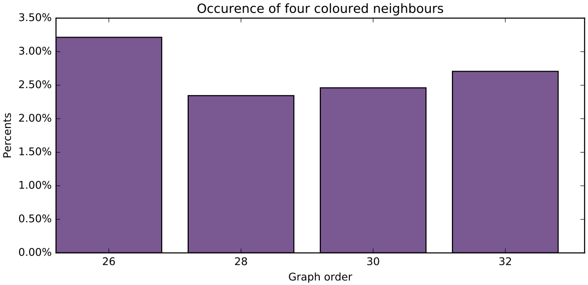

Question (4)

Mohli by ste vyskúšať (analyzovať aj implementovať) alternatívne stratégie usporiadania hrán na prehľadávanie (napr. najprv 2-faktor, najprv 1-faktor, prehľadávaním do šírky, náhodne)?

- Spomenuté stratégie sme neimplementovali:
- V prípade indexu, ktorým je daný graf ofarbiteľný, by alternatívne prístupy mohli pomôcť.
- Rozhodnutie, či daný index je pre graf indexom vyžaduje zamietnutie menších zlomkov
- príklad začneme 1-faktorom pri 23/7 farbeni: zložitosť $23^{|E|/3}$
- Poradie zvolené v práci odhaľuje lokálne konflikty čo najskôr. Malo by teda dosahovať lepšie výsledky (rýchlejšie časy).

Question (5)

Skúšali ste, či beh SAT-solvera závisí od usporiadania formúl?

- Nie, neskúšali, pretože v literatúre [MN14] sa uvádza, že náhodné preusporiadavanie formúl nemá vplyv na rýchlosť vyriešenia inštancie (viaceré SAT solvery, vrátane Lingeling).

Question (6)

Môžete spraviť odhad, koľko by trvalo vyrátať cirkulárny chromatický index snarkov na 32 prípadne 34 vrcholov?

- V práci je na strane 48 graf, v ktorom je priemer z desiatich 32 aj 34 vrcholových grafov. Výpočty v priemere trvali 3 minúty.

Question (7)

Aký je cirkulárny chromatický index cyklicky 5- a 6-súvislých snarkov (aj na viac ako 30 vrcholoch)?

- Priemerný čas behu pre jeden graf je 1m23.6s

Order	32
# snarks	2953
(7, 2)	0
(10, 3)	**753**
(11, 3)	0
(13, 4)	**1191**
(17, 5)	0
(23, 7)	**1009**
(29, 9)	0
(33, 10)	0