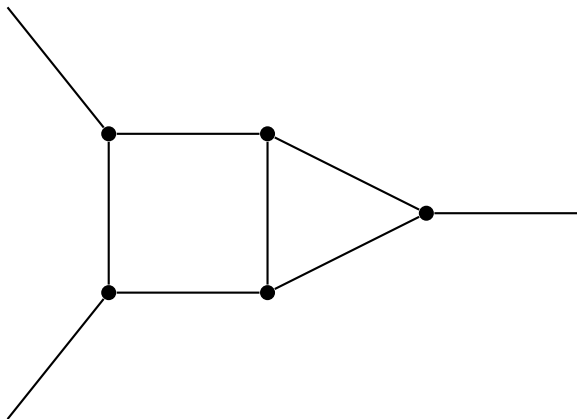


Tokové polynómy k -pólov

Diplomová práca

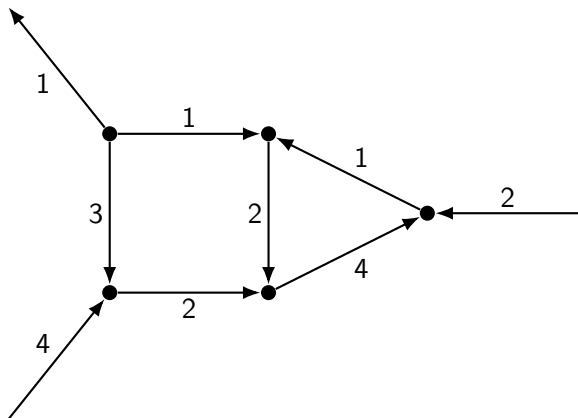
Bc. Dávid Mišiak
doc. RNDr. Robert Lukočka, PhD.

Katedra Informatiky
Fakulta Matematiky, Fyziky a Informatiky
Univerzita Komenského



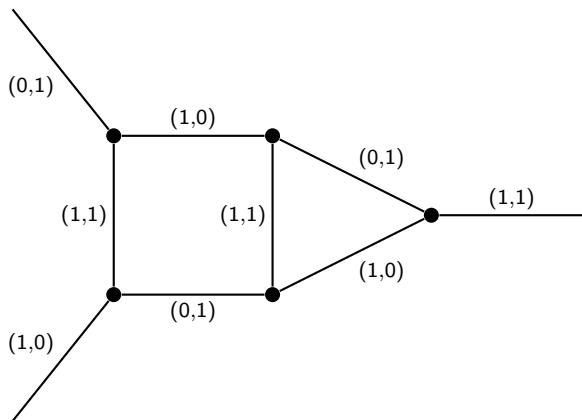
Obr.: Príklad kubického 3-pólu.

Nikde-nulový tok nad grupou

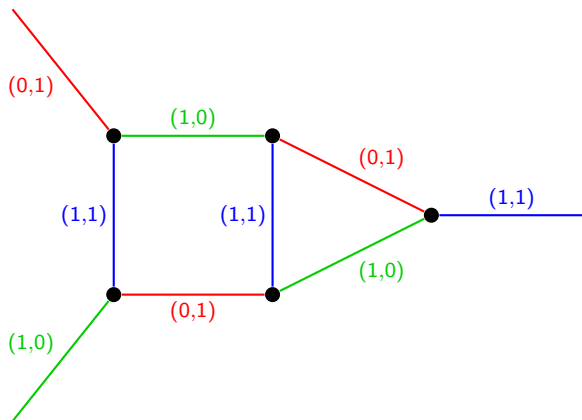


Obr.: Príklad nikde-nulového \mathbb{Z}_5 -toku v 3-póle.

$(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -tok



Obr.: Príklad nikde-nulového $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -toku v 3-póle. Na orientácii hrán nezáleží – v $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ je každý prvok involúciou.

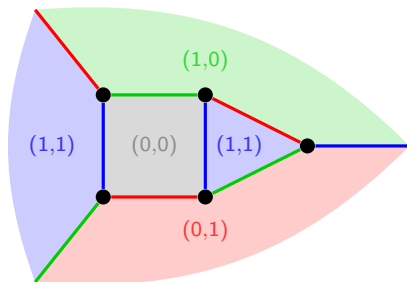


Obr.: Nikde-nulový $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -tok v kubickom k -póle tvorí hranové 3-farbenie.

Planárny k -pól G :

m -farbenie oblastí \Leftrightarrow nikde-nulový m -tok

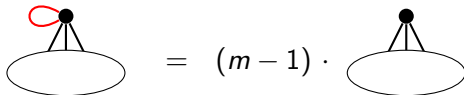
veta o 4 farbách \Leftrightarrow nikde-nulový $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -tok



+ zaujímavé sú kubické grafy

- ak e je slučka v G :

$$f(G) = (m - 1) \cdot f(G - e)$$



- inak:

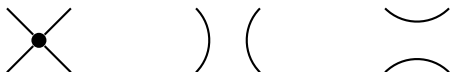
$$f(G) = f(G / e) - f(G - e)$$



Základné k -póly = bez vnútorných hrán



Obr.: Planárny 3-pól bez vnútorných hrán.



Obr.: Planárne 4-póly bez vnútorných hrán.



Obr.: Planárne 5-póly bez vnútorných hrán.

$n(G)$ ako kombinácia základných k -pólov

$$\begin{aligned} \text{Diagram 1} &= a \cdot \text{Diagram 2} + b_1 \cdot \text{Diagram 3} \quad \left(+ b_2 \cdot \text{Diagram 4} \right) = \\ &= 0 \cdot \text{Diagram 2} + 1 \cdot \text{Diagram 3} \quad \left(+ 1 \cdot \text{Diagram 4} \right) \end{aligned}$$

The diagram on the left is a square with four vertices, each having two external lines extending outwards. The first diagram in the first row is a vertex with four external lines crossing at a central point. The second diagram in the first row is a vertex with two external lines on the left and two on the right. The third diagram in the first row is a vertex with two external lines on the top and two on the bottom. The fourth diagram in the first row is a vertex with two external lines on the top and two on the bottom, with a different internal structure. The second row shows the same diagrams but with coefficients 0, 1, and 1 instead of a, b1, and b2.

$n(G)$ ako kombinácia základných k -pólov

$$\begin{aligned}
 \begin{array}{c} \diagup \text{red} \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \diagdown \text{green} \end{array} &= a \cdot \begin{array}{c} \diagup \text{red} \\ \bullet \\ \diagdown \text{green} \end{array} + b_1 \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) + b_2 \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \\
 &= 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0
 \end{aligned}$$

$n(G)$ ako kombinácia základných k -pólov

$$\begin{aligned}
 \begin{array}{c} \diagup \text{---} \diagdown \\ \bullet \text{---} \bullet \\ \diagdown \text{---} \diagup \\ \bullet \text{---} \bullet \\ \diagup \text{---} \diagdown \end{array} &= a \cdot \begin{array}{c} \diagup \text{---} \diagdown \\ \bullet \\ \diagdown \text{---} \diagup \end{array} + b_1 \cdot \begin{array}{c} \diagup \text{---} \diagdown \\ \text{---} \\ \diagdown \text{---} \diagup \end{array} + b_2 \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \\
 &= 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0
 \end{aligned}$$

Na \mathbb{Z}_5 :

Kochol (2005). Decomposition formulas for the flow polynomial.

Kochol (2005). Restrictions On Smallest Counterexamples To The 5-Flow Conjecture.

Kochol (2010). Smallest counterexample to the 5-flow conjecture has girth at least eleven.



X



X



✓

$$\text{circle with 4 lines} = a \cdot \text{circle with 4 lines and a dot} + b_1 \cdot \text{circle with 4 lines and a dot} + b_2 \cdot \text{circle with 4 lines and a dot}$$



$$a + b_1 + b_2 \geq 0$$



X

$$a + b_1 = 0$$



X

$$a + b_2 = 0$$



✓

$$a > 0$$

$$\text{Circle with 4 external lines} = a \cdot \text{Circle with 4 external lines and a central dot} + b_1 \cdot \text{Circle with 4 external lines and a dot on the top line} + b_2 \cdot \text{Circle with 4 external lines and a dot on the right line}$$



$$a + b_1 + b_2 \geq 0$$



X

$$a + b_1 = 0 \quad \Rightarrow \quad b_1 = -a$$



X

$$a + b_2 = 0 \quad \Rightarrow \quad b_2 = -a$$



✓

$$a > 0 \quad \Rightarrow \quad a + b_1 + b_2 < 0 \quad \text{SPOR}$$

$$\text{circle with 4 lines} = a \cdot \text{circle with 4 lines and a dot} + b_1 \cdot \text{circle with 4 lines and a dot} + b_2 \cdot \text{circle with 4 lines and a dot}$$



$$a + b_1 + b_2 \geq 0$$



X

$$a + b_1 = 0 \quad \Rightarrow \quad b_1 = -a$$



X

$$a + b_2 = 0 \quad \Rightarrow \quad b_2 = -a$$



✓

$$a > 0 \quad \Rightarrow \quad a + b_1 + b_2 < 0 \quad \text{SPOR}$$

k	koeficienty	farbenia
2	1	1
3	1	1
4	3	4
5	6	10
6	15	31
7	36	91
8	91	274
...	$O(3^k/k^{3/2})$	$O(3^k)$

→ pre planárne grafy sú koeficienty efektívnym nástrojom na štúdium farbení (pre neplanárne však nie)

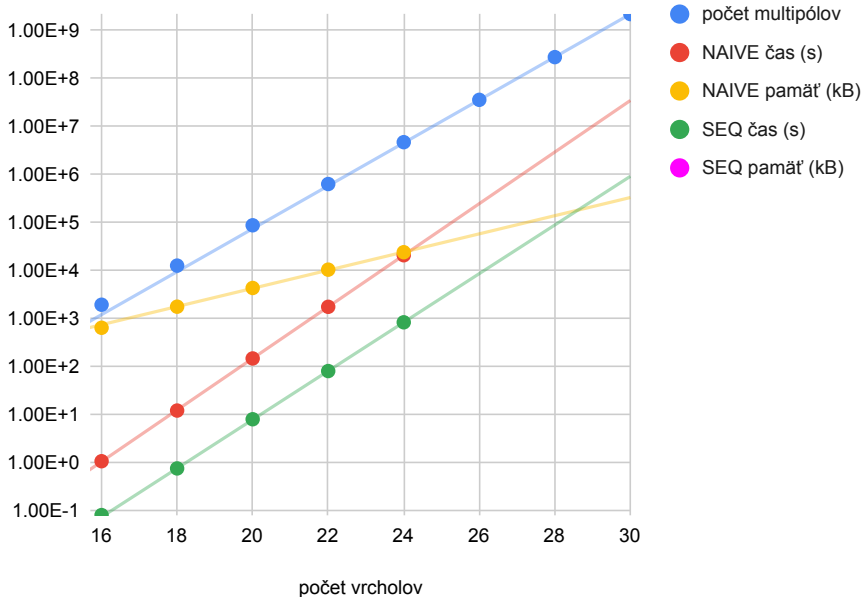
Naša práca

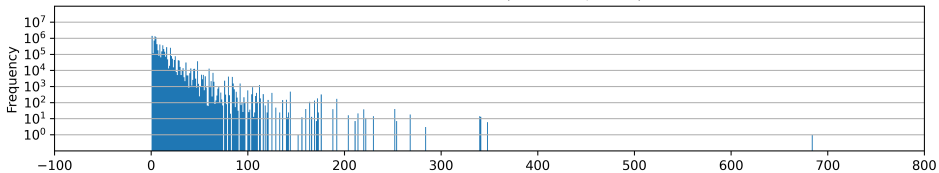
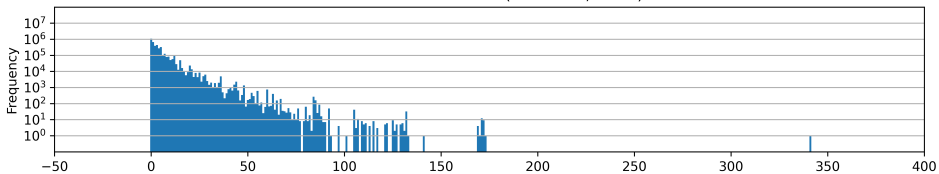
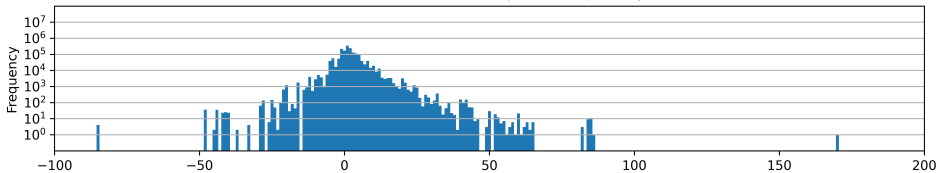
Algoritmus výpočtu koeficientov

- NAIVE: rekurzia, memoizácia, orezávanie

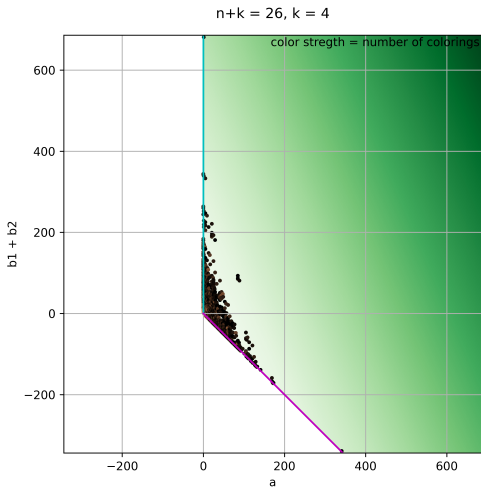
- SEQ:

The diagram illustrates the decomposition of a vertex with a loop into a sum of terms. On the left, a circle with four external legs and a loop on top is shown. A dashed line indicates the loop is cut, leading to a circle with four external legs. This is followed by an equals sign and a sum of terms: $a \cdot$ (crossed lines) $+ b_1 \cdot$ (right parenthesis) $(+ b_2 \cdot$ (left parenthesis). A downward arrow points to a sum of three terms: $a \cdot$ (two vertices connected by a loop) $+ b_1 \cdot$ (two vertices connected by a line) $+ b_2 \cdot$ (two vertices connected by a loop with a dot). A vertical ellipsis follows, and the final row shows $a' \cdot$ (crossed lines) $+ b'_1 \cdot$ (right parenthesis) $(+ b'_2 \cdot$ (left parenthesis).



star coefficient value ($n+k = 26$, $k = 3$)star coefficient value ($n+k = 26$, $k = 4$)star coefficient value ($n+k = 26$, $k = 5$)

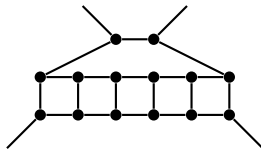
Teoretické a empirické obmedzenia ($k = 4$)



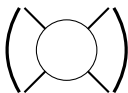
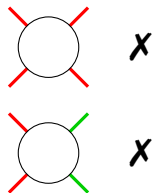
$$a \geq 0$$



$$a + b_1 + b_2 \geq 0$$

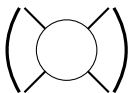
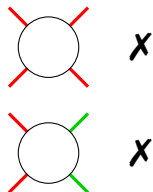


Dokreslenia na planárny snark ($k = 4$)



$$2a + 2b_1 + b_2 > 0$$

Dokreslenia na planárny snark ($k = 4$)

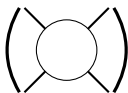
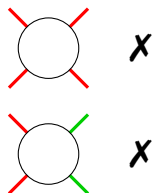


$$2a + 2b_1 + b_2 > 0$$

↕

G má most zľava-doprava

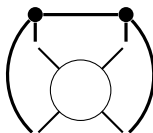
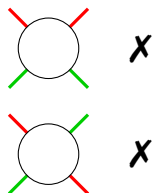
Dokreslenia na planárny snark ($k = 4$)



$$2a + 2b_1 + b_2 > 0$$

\Updownarrow

G má most zľava-doprava



$$2a + b_2 > 0$$

\Updownarrow

G je súvislý

Teoretické a empirické obmedzenia ($k = 5$)

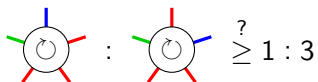
$$5a + 2b = \frac{1}{3}(5a + b) + \underbrace{\frac{5}{3}(2a + b)}_{\begin{matrix} ? \\ \geq 0 \end{matrix}}$$

$$\frac{5a + 2b}{5a + b} \stackrel{?}{\geq} \frac{1}{3}$$

Teoretické a empirické obmedzenia ($k = 5$)

$$5a + 2b = \frac{1}{3}(5a + b) + \underbrace{\frac{5}{3}(2a + b)}_{\geq 0}$$

$$\frac{5a + 2b}{5a + b} \stackrel{?}{\geq} \frac{1}{3}$$


$$: \stackrel{?}{\geq} 1 : 3$$

Hypotéza

Pre každý ofarbitel'ny planárny kubický 5-pól tvoria farbenia typu 00012 aspoň 1/4 celkového počtu farbení.

Teoretické a empirické obmedzenia ($k = 6$)



a



b



c

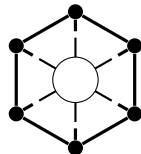


d



e

T	a	b	c	d	e	E
T_1	6	2	0	0	0	E_1
T_2	6	0	4	0	0	E_2
T_3	6	2	0	2	0	E_3
T_4	6	2	2	2	0	E_4
T_5	6	0	6	0	0	E_5
T_6	6	3	0	0	3	E_6
T_7	6	6	0	6	6	E_7
T_8	6	4	0	2	3	$\frac{1}{3}E_7 + \frac{2}{3}E_8$
T_9	6	1	2	0	0	$\frac{1}{2}E_1 + \frac{1}{2}E_8$
	6	3	0	0	$\frac{3}{2}$	E_8
	6	2	$\frac{4}{3}$	$\frac{2}{3}$	2	E_9
	6	3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	E_{10}



$\frac{1}{8}$ všetkých
farbení

Výsledky:

- koeficienty ako nástroj na štúdium farbení
- návrh a implementácia algoritmu na výpočet koeficientov
- výpočet koeficientov pre planárne kubické k -póly do ~ 30 vr.
- analýza teoretických a empirických obmedzení pre koeficienty
- hypotéza o minimálnom pomere počtu farbení v 5-póloch

Ďalší výskum:

- dôkaz hypotézy?
- systematickejšia analýza pre obmedzenia dané dokresleniami na planárny snark